Canonical reductions of principal bundles

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Abstract. We introduce the theory of vector bundles and principal bundles, and prove the existence and uniqueness of Harder-Narasimhan filtrations of vector bundles. Afterward, we generalize this result as canonical reductions of principal bundles.

We work with principal bundles whose structure groups are reductive groups, as seen in Chapter 1, studying the examples $\mathbf{GL}(r, \mathbb{C})$, $\mathbf{SL}(r, \mathbb{C})$, $\mathbf{SO}(r, \mathbb{C})$ and $\mathbf{Sp}(2n, \mathbb{C})$. Of importance are their Cartan, Borel and parabolic subgroups, and the root space decompositions of their Lie algebras.

In Chapter 2, we study degrees and slopes of vector bundles, as well as reductions of principal bundles. We then define slope-(semi)-stability of vector bundles and generalize this to Ramanathan-(semi)-stability of principal bundles.

In Chapter 3, we construct Harder-Narasimhan filtrations of vector bundles, and canonical reductions of principal bundles in two ways, proving that these are unique. We view examples of these filtrations for orthogonal bundles and symplectic bundles, and see how they store slope-semistability and Ramanathan-semistability properties.

In Chapter 4, we construct topological types of principal bundles, learning the obstruction theory to do so. We use this to succinctly characterize canonical reductions through Harder-Narasimhan types.

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Contents

CHAPTER 1

Parabolic subgroups of reductive groups

In this chapter, we follow [Hum72] to review complex reductive Lie algebras and their root space decompositions, and follow [Bor91] and [MT12] to review complex reductive groups.

Our examples include the general linear group $\mathbf{GL}(r,\mathbb{C})$, and for $r \geq 2$, also the special linear group $\mathbf{SL}(r,\mathbb{C})$, the special orthogonal group $\mathbf{SO}(r,\mathbb{C})$, the orthogonal group $\mathbf{O}(r,\mathbb{C})$ and the symplectic group $\mathbf{Sp}(2n,\mathbb{C})$, along with the Lie algebras of these groups.

In the end, we construct parabolic subgroups of these groups.

1.1. Reductive and semisimple groups

1.1.1. Reductive and semisimple Lie algebras

Let $\mathfrak{g} \neq 0$ be a finite-dimensional complex Lie algebra, with the Lie bracket $[\_,\_] : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. For subsets $A, B \subset \mathfrak{g}$, we write $[A,B] = \mathrm{span}_{\mathbb{C}}([X,Y] | X \in A, Y \in B)$.

Definition 1.1.1. (a) The Lie algebra $\mathfrak{g}$ is *abelian* if $[\mathfrak{g},\mathfrak{g}] = 0$.
(b) The Lie algebra $\mathfrak{g}$ is *semisimple* if $\mathfrak{g}$ has no nonzero abelian ideals.
(c) The Lie algebra $\mathfrak{g}$ is *reductive* if for every ideal $\mathfrak{a}$ of $\mathfrak{g}$, there exists a complementary ideal $\mathfrak{b}$ of $\mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ as Lie algebras.

In order to prove that certain Lie algebras are reductive or semisimple, we rely on characterizations of reductivity and semisimplicity using the following definitions.

Definition 1.1.2. (a) We define the *commutator series* of $\mathfrak{g}$:

$$\ldots \subseteq \mathfrak{g}^1 \subseteq \mathfrak{g}^0 = \mathfrak{g}, \tag{1.1.1}$$

such that $\mathfrak{g}^0 = \mathfrak{g}$, and for all $i \in \mathbb{N}$, we have $\mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}^{i-1}]$.
(b) The Lie algebra $\mathfrak{g}$ is *solvable* if there exists an $i \in \mathbb{N}$, such that $\mathfrak{g}^i = 0$.
(c) We define the *lower central series* of $\mathfrak{g}$:

$$\ldots \subseteq \mathfrak{g}_1 \subseteq \mathfrak{g}_0 = \mathfrak{g}, \tag{1.1.2}$$

such that $\mathfrak{g}_0 = \mathfrak{g}$, and for all $i \in \mathbb{N}$, we have $\mathfrak{g}_i = [\mathfrak{g}, \mathfrak{g}_{i-1}]$.
(d) The Lie algebra $\mathfrak{g}$ is *nilpotent* if there exists an $i \in \mathbb{N}$, such that $\mathfrak{g}_i = 0$.
(e) The *radical* $\mathfrak{r}(\mathfrak{g})$ of $\mathfrak{g}$ is the maximal solvable ideal of $\mathfrak{g}$.
(f) The *center* $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is the ideal $\{X \in \mathfrak{g} | \forall Y \in \mathfrak{g} : [X,Y] = 0\}$.

We also make use of Killing forms.

Definition 1.1.3. For all $X \in \mathfrak{g}$, we denote the adjoint representation of X by $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$. We define the *Killing form* $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ as the symmetric bilinear form:

$$\kappa(X,Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)). \tag{1.1.3}$$

The Killing form is ad-invariant, i.e., for all $X, Y, Z \in \mathfrak{g}$, we have:

$$\kappa(\mathrm{ad}(Z)(X), Y) = -\kappa(X, \mathrm{ad}(Z)(Y)). \tag{1.1.4}$$

For a complex Lie group G with the Lie algebra $\mathfrak{g}$, the Killing form is also Ad-invariant, i.e., for all $g \in G$ and for all $X, Y \in \mathfrak{g}$, we have:

$$\kappa(\mathrm{Ad}(g)(X), \mathrm{Ad}(g)(Y)) = \kappa(X, Y). \tag{1.1.5}$$

THEOREM 1.1.4. *The following are equivalent:*

(i) *The Lie algebra* $\mathfrak{g}$ *is semisimple.*
(ii) *We have* $\mathfrak{r}(\mathfrak{g}) = 0$.
(iii) *The Killing form* κ *is nondegenerate.*
(iv) *The Lie algebra* $\mathfrak{g}$ *decomposes into a direct sum of ideals that are simple Lie algebras.*

PROOF. The equivalence of *(i)* and *(ii)* can be verified directly. The equivalence of *(ii)* and *(iii)* is proven in [Kna88, I.7 Theorem 1.45], and the equivalence of *(ii)* and *(iv)* is proven in [Kna88, I.7 Theorem 1.54]. □

THEOREM 1.1.5. *The following are equivalent:*

(i) *The Lie algebra* $\mathfrak{g}$ *is reductive.*
(ii) *The Lie algebra* $\mathfrak{g}$ *is the direct sum* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$ *of a semisimple Lie algebra* $\mathfrak{s}$ *and an abelian Lie algebra* $\mathfrak{a}$.
(iii) *The adjoint representation ad* : $\mathfrak{g} \to \mathbf{Der}(\mathfrak{g})$ *is completely reducible, i.e., a direct sum of irreducible representations.*
(iv) *There exists a nondegenerate symmetric ad-invariant bilinear form on* $\mathfrak{g}$.
(v) *We have* $\mathfrak{r}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.
(vi) *For* $\mathfrak{g}_{ss} = [\mathfrak{g}, \mathfrak{g}]$, *we have* $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{z}(\mathfrak{g})$ *as Lie algebras.*

If these conditions are fulfilled, $\mathfrak{g}_{ss}$ *is semisimple.*

PROOF. The equivalence of *(i)* and *(ii)* is a consequence of Theorem 1.1.4.

The equivalence of *(ii)* and *(iii)* is clear, since irreducible subrepresentations of ad : $\mathfrak{g} \to \mathbf{Der}(\mathfrak{g})$ correspond ideals of $\mathfrak{g}$ that are simple Lie algebras.

The implication from *(ii)* to *(v)* is proven by directly calculating $\mathfrak{r}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$. For *(v)* implying *(ii)*, we verify $\mathfrak{g} \simeq (\mathfrak{g}/\mathfrak{r}(\mathfrak{g})) \oplus \mathfrak{r}(\mathfrak{g})$ as Lie algebras, where $\mathfrak{g}/\mathfrak{r}(\mathfrak{g})$ is semisimple due to [Kna88, I.3 Proposition 1.14], and $\mathfrak{r}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ is abelian.

We follow that *(ii)* implies *(iv)* by modifying the Killing form on $\mathfrak{g}$, and *(iv)* implies *(i)* through the use of complements $\mathfrak{a}^{\perp}$ of ideals $\mathfrak{a}$, with respect to the given bilinear form.

For *(ii)* being equivalent to *(vi)*, we use that $\mathfrak{g}_{ss} \simeq \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is semisimple. □

Due to this theorem, semisimple Lie algebras are reductive.

Our first example of a reductive Lie algebra is the Lie algebra $\mathfrak{gl}(r, \mathbb{C})$ of $\mathbf{GL}(r, \mathbb{C})$.

EXAMPLE 1.1.6. *The Lie algebra* $\mathfrak{gl}(r, \mathbb{C}) = Mat(r \times r, \mathbb{C})$ *of the complex algebraic group* $\mathbf{GL}(r, \mathbb{C}) = \{A \in Mat(r \times r, \mathbb{C}) | det(A) \neq 0\}$.

We have $\mathfrak{gl}(r, \mathbb{C}) \neq 0$. The bilinear form $\langle \_, \_ \rangle : \mathfrak{gl}(r, \mathbb{C}) \times \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$:

$$\langle X, Y \rangle = \mathrm{tr}(XY), \tag{1.1.6}$$

is symmetric, nondegenerate and ad-invariant. Hence, $\mathfrak{gl}(r, \mathbb{C})$ is reductive due to Theorem 1.1.5.

We denote the $r \times r$-identity matrix by I_r. By calculating that the center of $\mathbf{GL}(r, \mathbb{C})$ is $\mathbf{Z}(\mathbf{GL}(r, \mathbb{C})) = \{\lambda I_r | \lambda \in \mathbb{C}^{\times}\}$, we have its Lie algebra $\mathfrak{z}(\mathfrak{gl}(r, \mathbb{C})) = \mathrm{span}_{\mathbb{C}}(I_r)$, which is contained in $\mathfrak{r}(\mathfrak{gl}(r, \mathbb{C}))$. Thus, $\mathfrak{gl}(r, \mathbb{C})$ is not semisimple due to Theorem 1.1.4.

We now present examples of Lie subalgebras $\mathfrak{g} \neq 0$ of $\mathfrak{gl}(r, \mathbb{C})$ that are closed under conjugate transposition $(\_)^H$, and claim that they are reductive. In this case, $\mathfrak{g}$ is closed under the Cartan involution $\theta : \mathfrak{gl}(r, \mathbb{C}) \to \mathfrak{gl}(r, \mathbb{C})$:

$$X \mapsto -X^H, \tag{1.1.7}$$

as defined in [Kna88, VI.2]. Therefore, $\langle \_, \_ \rangle$ induces a Hermitian inner-product $\langle \_, \_ \rangle_\theta : \mathfrak{gl}(r, \mathbb{C}) \times \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$:

$$\langle X, Y \rangle_\theta = \operatorname{tr}(XY^H), \tag{1.1.8}$$

which restricts to a Hermitian inner-product on $\mathfrak{g}$, due to its positive-definiteness. Thus, $\langle \_, \_ \rangle$ restricts to a nondegenerate symmetric ad-invariant bilinear form on $\mathfrak{g}$, and $\mathfrak{g}$ is reductive due to Theorem 1.1.5.

The semisimplicity of $\mathfrak{g}$ is equivalent to $\mathfrak{z}(\mathfrak{g}) = 0$ due to Theorem 1.1.5.

EXAMPLE 1.1.7. (a) *The Lie algebra* $\mathfrak{sl}(r, \mathbb{C}) = \{X \in \mathfrak{gl}(r, \mathbb{C}) | tr(X) = 0\}$ *of the complex algebraic group* $\mathbf{SL}(r, \mathbb{C}) = \{A \in \mathbf{GL}(r, \mathbb{C}) | det(A) = 1\}$.

For $r = 1$, we have $\mathfrak{sl}(1, \mathbb{C}) = 0$.

For $r \geq 2$, we have $\mathfrak{sl}(r, \mathbb{C}) \neq 0$. The center $\mathbf{Z}(\mathbf{SL}(r, \mathbb{C}))$ is contained within $\mathbf{Z}(\mathbf{GL}(r, \mathbb{C}))$ and is thus finite, implying $\mathfrak{z}(\mathfrak{sl}(r, \mathbb{C})) = 0$. Thus, $\mathfrak{sl}(r, \mathbb{C})$ is semisimple.

For all $r \in \mathbb{N}$, as the Lie bracket is the commutator of matrices, $\mathfrak{gl}(r, \mathbb{C})_{ss}$ is trace-free, and $\mathfrak{gl}(r, \mathbb{C})_{ss} \subseteq \mathfrak{sl}(r, \mathbb{C})$. Since $\mathfrak{z}(\mathfrak{sl}(r, \mathbb{C})) = 0$, and since $\mathfrak{sl}(r, \mathbb{C})$ is an ideal of $\mathfrak{gl}(r, \mathbb{C})$, we also have $\mathfrak{sl}(r, \mathbb{C}) \subseteq \mathfrak{gl}(r, \mathbb{C})_{ss}$, implying $\mathfrak{gl}(r, \mathbb{C})_{ss} = \mathfrak{sl}(r, \mathbb{C})$.

(b) *The Lie algebra* $\mathfrak{so}(r, \mathbb{C}) = \{X \in \mathfrak{gl}(r, \mathbb{C}) | X + X^T = 0\}$ *of the complex algebraic group* $\mathbf{SO}(r, \mathbb{C}) = \{A \in \mathbf{SL}(r, \mathbb{C}) | A^T A = I_r\}$, *where* $(\_)^T$ *denotes transposition.*

For $r = 1$, we have $\mathfrak{so}(1, \mathbb{C}) = 0$.

For $r = 2$, we have $\mathfrak{so}(2, \mathbb{C}) \neq 0$. Furthermore, $\mathfrak{so}(2, \mathbb{C}) \simeq \mathbb{C}$ is an abelian Lie algebra, and is thus not semisimple, but still reductive.

For $r \geq 3$, we have $\mathfrak{so}(r, \mathbb{C}) \neq 0$. The center $\mathbf{Z}(\mathbf{SO}(r, \mathbb{C}))$ is finite, since $\mathbf{Z}(\mathbf{SO}(r, \mathbb{C}))$ is contained within the set of matrices λI_r, where $\lambda \in \mathbb{C}^\times$ is an r-th root of unity. Since the Lie algebra of $\mathbf{Z}(\mathbf{SO}(r, \mathbb{C}))$ is $\mathfrak{z}(\mathfrak{so}(r, \mathbb{C}))$, we have $\mathfrak{z}(\mathfrak{so}(r, \mathbb{C})) = 0$, and that $\mathfrak{so}(r, \mathbb{C})$ is semisimple.

Note that $\mathfrak{so}(r, \mathbb{C}) = \mathfrak{o}(r, \mathbb{C})$ is also the Lie algebra of the complex algebraic group $\mathbf{O}(r, \mathbb{C}) = \{A \in \mathbf{GL}(r, \mathbb{C}) | A^T A = I_r\}$.

(c) *The Lie algebra* $\mathfrak{sp}(2n, \mathbb{C}) = \{X \in \mathfrak{gl}(2n, \mathbb{C}) | J_{2n} X + X^T J_{2n} = 0\}$ *of the complex algebraic group* $\mathbf{Sp}(2n, \mathbb{C}) = \{A \in \mathbf{GL}(2n, \mathbb{C}) | A^T J_{2n} A = J_{2n}\}$, *where:*

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathbf{GL}(2n, \mathbb{C}). \tag{1.1.9}$$

We have $\mathfrak{sp}(2n, \mathbb{C}) \neq 0$. The center $\mathbf{Z}(\mathbf{Sp}(2n, \mathbb{C})) = \{I_{2n}, -I_{2n}\}$ is finite, and thus we have $\mathfrak{z}(\mathfrak{sp}(2n, \mathbb{C})) = 0$, and that $\mathfrak{sp}(2n, \mathbb{C})$ is semisimple.

1.1.2. Root space decompositions

We now cover root space decompositions of reductive Lie algebras $\mathfrak{g}$, with respect to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. These induce a root system $\Phi(\mathfrak{g}, \mathfrak{t})$ of a real subspace $V_\mathbb{R}$ of the dual space $\mathfrak{t}^\vee = \operatorname{Hom}_\mathbb{C}(\mathfrak{t}, \mathbb{C})$ of $\mathfrak{t}$.

For the rest of this chapter, let $\mathfrak{g}$ be a complex reductive Lie algebra.

DEFINITION 1.1.8. (a) An element $X \in \mathfrak{g}$ is *semisimple* if $\operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is semisimple or equivalently diagonalizable.

(b) A Lie subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is *toral* if all of its elements $X \in \mathfrak{t}$ are semisimple.

(c) A *Cartan subalgebra* $\mathfrak{t}$ of $\mathfrak{g}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$ that is self-normalizing, i.e., $\mathfrak{t} = \{X \in \mathfrak{g} | \forall Y \in \mathfrak{t} : [X, Y] \in \mathfrak{t}\}$.

REMARK 1.1.9. (a) There exists a bijection:

$$\{\text{Cartan subalgebras of } \mathfrak{g}\} \leftrightarrow \{\text{Cartan subalgebras of } \mathfrak{g}_{ss}\}, \tag{1.1.10}$$

$$\mathfrak{t} \mapsto \mathfrak{g}_{ss} \cap \mathfrak{t}, \tag{1.1.11}$$

$$\mathfrak{h} \oplus \mathfrak{z}(\mathfrak{g}) \leftarrow\!\shortmid \mathfrak{h}, \tag{1.1.12}$$

which is well-defined due to the decomposition $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{z}(\mathfrak{g})$ from Theorem 1.1.5. Analogous bijections hold for toral subalgebras and maximal toral subalgebras of $\mathfrak{g}$.

(b) A toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is an abelian subalgebra, following [Hum72, II Lemma 8.1] and (a).

(c) There exists a toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ that properly contains $\mathfrak{z}(\mathfrak{g})$, following [Hum72, IV 15.3] and (a).

For the rest of this chapter, let $\mathfrak{t}$ be a maximal toral subalgebra of the complex reductive Lie algebra $\mathfrak{g}$. For $\alpha \in \mathfrak{t}^\vee$, we define the *weight space* of α:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \forall Y \in \mathfrak{t} : \mathrm{ad}(Y)(X) = \alpha(Y)X\}. \tag{1.1.13}$$

We also define the set of *roots* $\Phi(\mathfrak{g}, \mathfrak{t}) = \{\alpha \in \mathfrak{t}^\vee | \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$.

The following lemma states how we can decompose $\mathfrak{g}$ into weight spaces.

Lemma 1.1.10. *There exists a weight space decomposition of* $\mathfrak{g}$*:*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})} \mathfrak{g}_\alpha, \tag{1.1.14}$$

as a direct sum of complex vector spaces.

Proof. See [Hum72, II 8.1] and (a) of Remark 1.1.9. □

Remark 1.1.11. In the situation of Lemma 1.1.10, we have:

(a) For $0 \in \mathfrak{t}^\vee$, we have $\mathfrak{t} = \mathfrak{g}_0$, as proven in [Hum72, II Proposition 8.2].

(b) For all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$, we have $\dim_{\mathbb{C}}(\mathfrak{g}_\alpha) = 1$, as proven in [Hum72, II Proposition 8.4].

(c) For all $\alpha, \beta \in \mathfrak{t}^\vee$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{ss} \cap \mathfrak{g}_{\alpha+\beta}$, following from [Hum72, II Proposition 8.4] and (a) of Remark 1.1.9.

(d) The bijection in (a) of Remark 1.1.9 induces a bijection:

$$\Phi(\mathfrak{g}, \mathfrak{t}) \to \Phi(\mathfrak{g}_{ss}, \mathfrak{g}_{ss} \cap \mathfrak{t}), \quad \alpha \mapsto \alpha|_{\mathfrak{g}_{ss} \cap \mathfrak{t}}, \tag{1.1.15}$$

that preserves weight spaces, i.e., $\mathfrak{g}_\alpha = (\mathfrak{g}_{ss})_{\alpha|_{\mathfrak{g}_{ss} \cap \mathfrak{t}}}$.

Note that in Lemma 1.1.11, we can equivalently refer to $\mathfrak{t}$ as a Cartan subalgebra due to the following.

Lemma 1.1.12. *For a Lie subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$*, the following are equivalent:*

(i) The subalgebra $\mathfrak{h}$ *is a maximal toral subalgebra of* $\mathfrak{g}$*.*

(ii) The subalgebra $\mathfrak{h}$ *is a Cartan subalgebra of* $\mathfrak{g}$*.*

Proof. See [Hum72, IV Corollary 15.3] and (a) of Remark 1.1.9. □

Altogether, $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, where there exists a weight space decomposition:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})} \mathfrak{g}_\alpha. \tag{1.1.16}$$

We now wish to confirm that $\Phi(\mathfrak{g}, \mathfrak{t})$ forms a root system. Firstly, it is shown in [Hum72, II Corollary 8.2] that the Killing form κ restricted to $\mathfrak{g}_{ss} \cap \mathfrak{t}$ is nondegenerate, inducing an isomorphism of complex vector spaces:

$$(\mathfrak{g}_{ss} \cap \mathfrak{t})^\vee \to \mathfrak{g}_{ss} \cap \mathfrak{t}, \quad \alpha \mapsto H_\alpha, \quad \text{such that } \alpha = \kappa(\_, H_\alpha). \tag{1.1.17}$$

Through this, κ induces a nondegenerate symmetric bilinear form $(\_, \_) : (\mathfrak{g}_{ss} \cap \mathfrak{t})^\vee \times (\mathfrak{g}_{ss} \cap \mathfrak{t})^\vee \to \mathbb{C}$:

$$(\alpha, \beta) = \kappa(H_\alpha, H_\beta). \tag{1.1.18}$$

THEOREM 1.1.13. *There exists a natural isomorphism of complex vector spaces between* $(\mathfrak{g}_{ss} \cap \mathfrak{t})^\vee$ *and* $V = span_{\mathbb{C}}(\Phi(\mathfrak{g}, \mathfrak{t}))$*, such that* $(\_, \_)$ *restricts to a symmetric inner-product* $\langle \_, \_ \rangle$ *on* $V_{\mathbb{R}} = span_{\mathbb{R}}(\Phi(\mathfrak{g}, \mathfrak{t}))$*.*

The set $\Phi(\mathfrak{g}, \mathfrak{t})$ *is a root system of* $(V_{\mathbb{R}}, \langle \_, \_ \rangle)$*, i.e.:*

(i) The set $\Phi(\mathfrak{g}, \mathfrak{t})$ *is finite, and does not contain* 0*.*
(ii) For all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$*, the only integer multiples of* α *in* $\Phi(\mathfrak{g}, \mathfrak{t})$ *are* $\alpha, -\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$*.*

For all nonzero $\alpha \in V_{\mathbb{R}}$*, we define the reflection* $s_\alpha : V_{\mathbb{R}} \to V_{\mathbb{R}}$*:*

$$s_\alpha(x) = x - 2\frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha. \tag{1.1.19}$$

(iii) For all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$*, we have* $s_\alpha(\Phi(\mathfrak{g}, \mathfrak{t})) \subseteq \Phi(\mathfrak{g}, \mathfrak{t})$*.*
(iv) For all $\alpha, \beta \in \Phi(\mathfrak{g}, \mathfrak{t})$*,* $s_\alpha(\beta) - \beta$ *is an integer multiple of* α*.*

PROOF. See [Hum72, II Theorem 8.5] and (a) of Remark 1.1.9. □

Since we know that $\Phi(\mathfrak{g}, \mathfrak{t})$ is a root system, we call the decomposition in (1.1.16) the *root space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Note that $\Phi(\mathfrak{g}, \mathfrak{t})$ and $\Phi(\mathfrak{g}_{ss}, \mathfrak{g}_{ss} \cap \mathfrak{t})$ define root systems of the same type, in terms of the Dynkin classification from [Hum72, III Theorem 11.4].

We now define Borel subalgebras and parabolic subalgebras of $\mathfrak{g}$, and find characterizations of them using that $\Phi(\mathfrak{g}, \mathfrak{t})$ is a root system.

DEFINITION 1.1.14. (a) A *Borel subalgebra* $\mathfrak{b}$ of $\mathfrak{g}$ is a maximal solvable Lie subalgebra of $\mathfrak{g}$.
(b) A *parabolic subalgebra* $\mathfrak{p}$ of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ containing a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$.
(c) A parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is *maximal*, if it is maximal in terms of inclusion amongst all proper parabolic subalgebras of $\mathfrak{g}$.

In order to characterize all Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{t}$, we introduce positive roots, as defined in [Hum72, III 10.1].

DEFINITION 1.1.15. A choice of half the roots $\Phi(\mathfrak{g}, \mathfrak{t})^+ \subseteq \Phi(\mathfrak{g}, \mathfrak{t})$ is a set of *positive roots* of $\Phi(\mathfrak{g}, \mathfrak{t})$ if we have the following:

(i) For all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$, we have $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})^+$ or $-\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})^+$, but not both.
(ii) For all $\alpha, \beta \in \Phi(\mathfrak{g}, \mathfrak{t})^+$ such that $\alpha + \beta \in \Phi(\mathfrak{g}, \mathfrak{t})$, we have $\alpha + \beta \in \Phi(\mathfrak{g}, \mathfrak{t})^+$.

REMARK 1.1.16. The following are in correspondence:

(a) Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{t}$.
(b) Subsets of positive roots $\Phi(\mathfrak{g}, \mathfrak{t})^+ \subseteq \Phi(\mathfrak{g}, \mathfrak{t})$.

As seen in [Hum72, IV 16.3], this correspondence is found through the decomposition:

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})^+} \mathfrak{g}_\alpha. \tag{1.1.20}$$

In order to similarly characterize parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$, we need to know that positive roots induce simple roots, as shown in [Hum72, III Theorem' 10.1].

REMARK 1.1.17. Given positive roots $\Phi(\mathfrak{g}, \mathfrak{t})^+ \subseteq \Phi(\mathfrak{g}, \mathfrak{t})$, there exists a unique subset $\triangle \subseteq \Phi(\mathfrak{g}, \mathfrak{t})^+$ of *simple roots*, such that $\triangle$ fulfills the following:

(i) Every root in $\Phi(\mathfrak{g}, \mathfrak{t})^+$ is a nonnegative integer linear combination of roots in $\triangle$.
(ii) The set $\triangle$ is minimal amongst all subsets of $\Phi(\mathfrak{g}, \mathfrak{t})^+$ fulfilling (i), with respect to inclusion. Equivalently, $\triangle$ forms an $\mathbb{R}$-basis of $V_{\mathbb{R}}$.

Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$, corresponding to $\Phi(\mathfrak{g}, \mathfrak{t})^+$ in the sense of Remark 1.1.16.

REMARK 1.1.18. The following are in correspondence:

(a) Parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{b}$.
(b) Subsets I of the simple roots $\triangle$ of $\Phi(\mathfrak{g},\mathfrak{t})^+$.

Such parabolic subalgebras in (a) are called *standard*.

As seen [Hum72, IV Exercise 16.6], this correspondence is found through the decomposition:

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Gamma_I} \mathfrak{g}_\alpha, \quad \Gamma_I = \Phi(\mathfrak{g},\mathfrak{t})^+ \cup \{\alpha \in \Phi(\mathfrak{g},\mathfrak{t}) | \alpha \in \operatorname{span}_{\mathbb{Z}}(\triangle \setminus I)\}. \tag{1.1.21}$$

Due to this, we write $\mathfrak{p}_I$ for $\mathfrak{p}$. We also have the following useful equivalences:

(c) The inclusions $I \subseteq J \subseteq \triangle$ are equivalent to $\mathfrak{p}_J \subseteq \mathfrak{p}_I$.
(d) The equality $I = \emptyset$ is equivalent to $\mathfrak{p}_I = \mathfrak{g}$.
(e) The equality $I = \triangle$ is equivalent to $\mathfrak{p}_I = \mathfrak{b}$.
(f) The set $I = \{*\}$ being a singleton is equivalent to $\mathfrak{p}_I$ being a maximal standard parabolic subgroup of $\mathfrak{g}$.

Using these results, we can identify all Cartan subalgebras of $\mathfrak{g}$ with $\mathfrak{t}$ and all Borel subalgebras of $\mathfrak{g}$ with $\mathfrak{b}$.

REMARK 1.1.19. Borel subalgebras of $\mathfrak{g}$ are conjugate to each other, and the same holds for Cartan subalgebras of $\mathfrak{g}$. This follows from [Hum72, IV 16.4]. Thus, all parabolic subalgebras of $\mathfrak{g}$ are conjugate to standard parabolic subalgebras of $\mathfrak{g}$.

This remark allows use define ranks of $\mathfrak{g}$.

DEFINITION 1.1.20. (a) The *rank* of $\mathfrak{g}$ is the dimension $\dim_{\mathbb{C}}(\mathfrak{t})$ of any Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$.
(b) The *semisimple rank* of $\mathfrak{g}$ is the dimension $\dim_{\mathbb{C}}(\mathfrak{g}_{ss}\cap\mathfrak{t})$ for any Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$.

The semisimple rank is also equal to the number of elements in $\triangle$. If $\mathfrak{g}$ is semisimple, its rank and semisimple rank coincides. However, if $\mathfrak{g}$ is reductive but not semisimple, the semisimple rank of $\mathfrak{g}$ is less than the rank of $\mathfrak{g}$.

Finally, we mention Levi decompositions of standard parabolic subalgebras of $\mathfrak{g}$.

REMARK 1.1.21. For a standard parabolic subalgebra $\mathfrak{p}_I$ of $\mathfrak{g}$, corresponding to $I \subseteq \triangle$, we have the *Levi decomposition* $\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$ as Lie algebras, given by:

$$\mathfrak{l}_I = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Gamma_I\cap-\Gamma_I} \mathfrak{g}_\alpha, \quad \mathfrak{u}_I = \bigoplus_{\alpha\in\Gamma_I\setminus-\Gamma_I} \mathfrak{g}_\alpha. \tag{1.1.22}$$

The *Levi-factor* $\mathfrak{l}_I$ is reductive and the *nilpotent radical* $\mathfrak{u}_I$ is nilpotent.

1.1.3. Reductive and semisimple groups

We now construct complex reductive and semisimple groups, whose Lie algebras are reductive and semisimple Lie algebras, following [Bor91] and [MT12]. We will see that these constructions group-theoretic analogs of those in Subsection 1.1.1 and Subsection 1.1.2.

Let G be a complex linear algebraic group, carrying the Zariski topology, with neutral element $e \in G$ and Lie algebra $\mathfrak{g} \neq 0$.

DEFINITION 1.1.22. (a) The *radical* $\mathbf{R}(G)$ of G is the maximal closed connected solvable complex algebraic normal subgroup of G, where solvable groups are defined as in [Bor91, I.2.4].
(b) The *unipotent radical* $\mathbf{R}_u(G)$ of G is the maximal closed connected unipotent complex algebraic normal subgroup of G, where unipotent groups are linear algebraic groups whose elements are unipotent, as defined in [MT12, Definition 2.6].

(c) The group G is called *semisimple* if $\mathbf{R}(G) = e$.
(d) The group G is called *reductive* if $\mathbf{R}_u(G) = e$.

Since $\mathbf{R}_u(G)$ consists of the unipotent elements of $\mathbf{R}(G)$, semisimple groups are reductive.

DEFINITION 1.1.23. (a) A *Borel* subgroup B of G is a maximal closed connected solvable complex algebraic subgroup of G.
(b) A *parabolic* subgroup P of G is a closed complex algebraic subgroup of G containing a Borel subgroup B of G.
(c) A parabolic subgroup P of G is *maximal*, if it is maximal in terms of inclusion amongst all proper parabolic subgroups of G.
(d) If a Borel subgroup B is fixed, then P is called *standard* if it contains B.
(e) A *Cartan* subgroup T of G is the centralizer of a maximal torus of G.

An equivalent condition for parabolic subgroups P of G is that it is a closed complex algebraic subgroup such that G/P is a complex complete algebraic variety, using [Bor91, IV.11.2 Corollary] and [MT12, Theorem 6.4]. As a group-theoretic analog of Lemma 1.1.12, if G is reductive, Cartan subgroups are equivalently maximal tori, as shown in [Bor91, IV.13.17 Corollary 2].

In order to determine whether certain complex linear algebraic groups are semisimple or reductive, we make use of the following characterizations of radicals.

REMARK 1.1.24. (a) From [MT12, Proposition 6.16], we have that:

$$\mathbf{R}(G) = \left(\bigcap_{B \subseteq G \text{ is a Borel subgroup}} B \right)^0, \tag{1.1.23}$$

where $(\_)^0$ denotes the identity component of a complex algebraic group.
(b) If G is reductive, then $\mathbf{R}(G) = \mathbf{Z}(G)^0$, as seen in [Bor91, IV.11.21 Proposition].

REMARK 1.1.25. (a) In [MT12, Theorem 8.17], it is proven that the Lie algebras of semisimple groups are semisimple. Analogous statements also hold for reductive groups, and their Cartan, Borel and (standard) parabolic subgroups.
(b) If G is reductive, the exponential map $\exp_G : \mathfrak{g} \to G$, as defined in [Čap23, 1.8], provides a bijection:

$$\{\text{Parabolic subalgebras of } \mathfrak{g}\} \leftrightarrow \{\text{Connected parabolic subgroups of } G\}. \tag{1.1.24}$$

Analogous bijections hold for Cartans and Borels of $\mathfrak{g}$ and G. Furthermore, when we fix a Borel subgroup B of G, an analogous bijection also holds for connected standard parabolics.

By further fixing a Cartan subgroup T of G, contained within B, we can identify connected standard parabolic subgroups P of G with subsets $I \subseteq \triangle$, writing P_I for P, where the Lie algebra of P_I is $\mathfrak{p}_I$, from Remark 1.1.18.

Note that if G is connected, all parabolic subgroups are connected, as seen in [Bor91, IV.11.16 Theorem].

Similarly to Remark 1.1.19, these correspondences imply the conjugacy of Cartan, Borel and parabolic subgroups of G.

REMARK 1.1.26. Borel subgroups of G are conjugate to each other, and the same holds for Cartan subgroups of G. This follows from [Bor91, IV.11.1 Theorem, IV.11.3 Corollary]. Thus, all parabolic subgroups of G are conjugate to standard parabolic subgroups of G.

Finally, we mention Levi decompositions of standard parabolic subgroups of a complex reductive group G, with respect to a Cartan subgroup T of G, and a Borel subgroup B of G containing T.

Let P_I be a standard parabolic subgroup of G, corresponding to $I \subseteq \triangle$.

Remark 1.1.27. The *unipotent radical* of P_I is $U_I = \mathbf{R}_u(P_I)$, which has the Lie algebra $\mathfrak{u}_I$, from the Levi decomposition in Remark 1.1.21.

There exists a unique closed complex reductive subgroup L_I of P_I, called the *Levi-factor* of P_I, containing T, such that $L_I \simeq P_I/U_I$, and such that the Lie algebra of L_I is $\mathfrak{l}_I$.

As complex algebraic groups, it is shown in [Bor91, IV.14.18 Proposition, IV.14.19 Corollary] that $P_I \simeq U_I \rtimes L_I$ is a semi-direct product of U_I and L_I. We call $U_I \rtimes L_I$ the *Levi decomposition* of P_I.

We now present examples of complex reductive and semisimple groups, whose Lie algebras are those from Example 1.1.6 and Example 1.1.7.

Example 1.1.28. (a) *The connected complex algebraic group* $\mathbf{GL}(r, \mathbb{C})$.

We have $\mathfrak{gl}(r, \mathbb{C}) \neq 0$. The Lie-Kolchin theorem, from [MT12, Theorem 4.1], applied on the canonical representation $\rho : \mathbf{GL}(r, \mathbb{C}) \to \mathbf{GL}(\mathbb{C}^r)$, implies that any solvable complex algebraic subgroup is the stabilizer of a flag of subspaces of $\mathbb{C}^r$. Thus, the upper-right triangular matrices of $\mathbf{GL}(r, \mathbb{C})$ form a Borel subgroup of $\mathbf{GL}(r, \mathbb{C})$.

Using another representation of $\mathbf{GL}(r, \mathbb{C})$, we also have that the lower-left triangular matrices of $\mathbf{GL}(r, \mathbb{C})$ form a Borel subgroup of $\mathbf{GL}(r, \mathbb{C})$. Using (a) of Remark 1.1.24, we intersect these Borel subgroups and see that $\mathbf{R}(\mathbf{GL}(r, \mathbb{C}))$ is contained within the diagonal matrices of $\mathbf{GL}(r, \mathbb{C})$. Thus, $\mathbf{R}_u(\mathbf{GL}(r, \mathbb{C})) = I_r$, and $\mathbf{GL}(r, \mathbb{C})$ is reductive.

Using (b) of Remark 1.1.24, $\mathbf{GL}(r, \mathbb{C})$ is not semisimple, since $\mathbf{R}(\mathbf{GL}(r, \mathbb{C})) = \mathbf{Z}(\mathbf{GL}(r, \mathbb{C}))^0$ is nontrivial.

(b) *The connected complex algebraic group* $\mathbf{SL}(r, \mathbb{C})$.

For $r = 1$, we have $\mathfrak{sl}(1, \mathbb{C}) = 0$.

For $r \geq 2$, we have $\mathfrak{sl}(r, \mathbb{C}) \neq 0$. The same argument as in (a) gives us that $\mathbf{R}_u(\mathbf{SL}(r, \mathbb{C})) = I_r$. Thus, $\mathbf{SL}(r, \mathbb{C})$ is reductive. Since $\mathbf{R}(\mathbf{SL}(r, \mathbb{C})) = \mathbf{Z}(\mathbf{SL}(r, \mathbb{C}))^0 = I_r$, $\mathbf{SL}(r, \mathbb{C})$ is semisimple.

(c) *The connected complex algebraic group* $\mathbf{SO}(r, \mathbb{C})$.

For $r = 1$, we have $\mathfrak{so}(1, \mathbb{C}) = 0$.

For $r = 2$, we have $\mathfrak{so}(2, \mathbb{C}) \neq 0$. Furthermore, $\mathbf{SO}(2, \mathbb{C}) \simeq \mathbb{C}^\times$ is abelian, and is thus not semisimple, but still reductive.

For $r \geq 3$, we have $\mathfrak{so}(r, \mathbb{C}) \neq 0$. By using the Lie-Kolchin theorem, we can use a similar argument to (a) to follow that $\mathbf{R}_u(\mathbf{SO}(r, \mathbb{C})) = I_r$. Thus, $\mathbf{SO}(r, \mathbb{C})$ is reductive. Since $\mathbf{R}(\mathbf{SO}(r, \mathbb{C})) = \mathbf{Z}(\mathbf{SO}(r, \mathbb{C}))^0 = I_r$, $\mathbf{SO}(r, \mathbb{C})$ is semisimple.

(d) *The unconnected complex algebraic group* $\mathbf{O}(r, \mathbb{C})$. Since the identity component of $\mathbf{O}(r, \mathbb{C})$ is $\mathbf{SO}(r, \mathbb{C})$, the same conclusions from (c) for $\mathbf{SO}(r, \mathbb{C})$ apply to $\mathbf{O}(r, \mathbb{C})$.

(e) *The connected complex algebraic group* $\mathbf{Sp}(2n, \mathbb{C})$.

We have $\mathfrak{sp}(2n, \mathbb{C}) \neq 0$. By using the Lie-Kolchin theorem, we can use a similar argument to (a) to follow that $\mathbf{R}_u(\mathbf{Sp}(2n, \mathbb{C})) = I_{2n}$. Thus, $\mathbf{Sp}(2n, \mathbb{C})$ is reductive. Since $\mathbf{R}(\mathbf{Sp}(2n, \mathbb{C})) = \mathbf{Z}(\mathbf{Sp}(2n, \mathbb{C}))^0 = I_{2n}$, $\mathbf{Sp}(2n, \mathbb{C})$ is semisimple.

1.2. Parabolic subgroups of reductive groups

Having introduced examples of complex reductive and semisimple Lie algebras and groups, we can now describe their root space decompositions and standard parabolics.

For each connected complex reductive group G in our list, with the Lie algebra $\mathfrak{g}$:

(Step 1) We find a suitable Cartan subgroup T of G, and its Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, and determine the rank and semisimple rank of $\mathfrak{g}$.

(Step 2) We find the roots $\Phi(\mathfrak{g},\mathfrak{t})$, and the root space decomposition of $\mathfrak{g}$. Then we find a suitable choice of positive roots $\Phi(\mathfrak{g},\mathfrak{t})^+$, and the induced simple roots $\triangle$. We also categorize the root system $\Phi(\mathfrak{g},\mathfrak{t})$, in the sense of the Dynkin classification from [Hum72, III Theorem 11.4].

(Step 3) We find the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and the Borel subgroup B of G, corresponding to $\Phi(\mathfrak{g},\mathfrak{t})^+$. We find the standard parabolic subalgebras $\mathfrak{p}_I$ of $\mathfrak{g}$, the standard parabolic subgroups P_I of G, and how they correspond to subsets $I \subseteq \triangle$.

(Step 4) We find the Levi decompositions $\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$ and $P_I \simeq U_I \rtimes L_I$.

1.2.1. The cases of $\mathbf{SL}(r,\mathbb{C})$ and $\mathbf{GL}(r,\mathbb{C})$

Example 1.2.1. *The connected complex semisimple group* $\mathbf{SL}(r,\mathbb{C})$*,* $r \geq 2$*.*

(Step 1) For $i,j = 1,\dots,r$, let E_{ij} be the $r \times r$-matrix with 1 on the i-th row and j-th column, and 0 elsewhere, and for $i = 1,\dots,r-1$, let $T_i = E_{ii} - E_{i+1,i+1}$. Then $\mathfrak{sl}(r,\mathbb{C})$ has the $\mathbb{C}$-basis $(E_{ij}, T_1,\dots,T_{r-1} | i,j = 1,\dots,r : i \neq j)$.

We claim that the subalgebra $\mathfrak{t} = \operatorname{span}_{\mathbb{C}}(T_1,\dots T_r)$ of diagonal matrices is a Cartan subalgebra of $\mathfrak{sl}(r,\mathbb{C})$. We calculate for all $k,l = 1,\dots,r-1$, and all $i,j = 1,\dots,r$, $i \neq j$, that:

$$[T_k, E_{k,j}] = E_{k,j}, \qquad j \neq k, k+1, \tag{1.2.1}$$
$$[T_k, E_{k+1,j}] = -E_{k+1,j}, \qquad j \neq k, k+1, \tag{1.2.2}$$
$$[T_k, E_{i,k}] = -E_{i,k}, \qquad i \neq k, k+1, \tag{1.2.3}$$
$$[T_k, E_{i,k+1}] = E_{i,k+1}, \qquad i \neq k, k+1, \tag{1.2.4}$$
$$[T_k, E_{k,k+1}] = 2E_{k,k+1}, \tag{1.2.5}$$
$$[T_k, E_{k+1,k}] = -2E_{k+1,k}, \tag{1.2.6}$$
$$[T_k, E_{ij}] = 0, \qquad i \neq k, k+1 \text{ and } j \neq k, k+1, \tag{1.2.7}$$
$$[T_k, T_l] = 0. \tag{1.2.8}$$

The subalgebra $\mathfrak{t}$ is thus abelian and nilpotent, and the calculations (1.2.1)...(1.2.8) imply that $\mathfrak{t}$ is self-normalizing, i.e., $\mathfrak{t} = \{X \in \mathfrak{sl}(r,\mathbb{C}) | \forall Y \in \mathfrak{t} : [X,Y] \in \mathfrak{t}\}$. Thus, $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{sl}(r,\mathbb{C})$, which has rank $\dim_{\mathbb{C}}(\mathfrak{t}) = r-1$, equal to its semisimple rank as well.

Therefore, the corresponding subgroup of diagonal matrices T of $\mathbf{SL}(r,\mathbb{C})$ is a maximal torus isomorphic to $(\mathbb{C}^\times)^{r-1}$, and thus a Cartan subgroup.

(Step 2) We have that $(E_{11},\dots,E_{rr})$ forms a $\mathbb{C}$-basis of the diagonal matrices of $\mathfrak{gl}(r,\mathbb{C})$. For the induced dual $\mathbb{C}$-basis $(E_{11}^\vee,\dots,E_{rr}^\vee)$ and $i = 1,\dots,r$, we denote $e_i = E_{ii}^\vee|_{\mathfrak{t}} : \mathfrak{t} \to \mathbb{C}$. Then for all $i,j = 1,\dots,r$, $i \neq j$, we define $\alpha_{ij} = e_i - e_j \in \mathfrak{t}^\vee$.

For all $i,j = 1,\dots,r$, $i \neq j$, and all $T \in \mathfrak{t}$, the calculations (1.2.1)...(1.2.8) can be summarized as $[T, E_{ij}] = \alpha_{ij}(T)E_{ij}$. Thus, the roots are:

$$\Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t}) = \{\alpha_{ij} | i,j = 1,\dots,r : i \neq j\}, \tag{1.2.9}$$

with the weight spaces $\mathfrak{sl}(r,\mathbb{C})_{\alpha_{ij}} = \operatorname{span}_{\mathbb{C}}(E_{ij})$, $i,j = 1,\dots,r$, $i \neq j$, such that the root space decomposition is:

$$\mathfrak{sl}(r,\mathbb{C}) = \mathfrak{t} \oplus \bigoplus_{\alpha_{ij} \in \Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t})} \operatorname{span}_{\mathbb{C}}(E_{ij}). \tag{1.2.10}$$

We claim that the set $\Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t})^+ = \{\alpha_{ij} | i,j = 1,\dots,r : i < j\}$ forms positive roots of $\Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t})$, corresponding to the weight spaces spanned by E_{ij}, where the nontrivial entry 1 lies above or right of the diagonal.

Firstly, it is clear that for all $\alpha_{ij} \in \Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t})$, we have that either $\alpha_{ij} \in \Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t})^+$ or $-\alpha_{ij} = \alpha_{ji} \in \Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t})^+$, since $i \neq j$. Furthermore, if $\alpha_{ij}, \alpha_{kl} \in \Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t})^+$, such that $\alpha_{ij} + \alpha_{kl} \in \Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t})$, we have $j = k$ and $\alpha_{ij} + \alpha_{kl} = \alpha_{il} \in \Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t})^+$, since $i < j = k < l$ implies $i < l$.

The simple roots are $\triangle = \{\alpha_{i,i+1} | i = 1, \ldots, r-1\}$, since $\triangle$ spans $\mathfrak{t}^\vee$, and for all $\alpha_{ij} \in \Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t})^+$, we have that $\alpha_{ij} = \alpha_{i,i+1} + \ldots + \alpha_{j-1,j}$ is a nonnegative linear combination of roots in $\triangle$.

In the sense of the Dynkin classification from [Hum72, III Theorem 11.4], $\Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{t})$ is a root system of type A_{r-1}.

(Step 3) The induced Borel subalgebra $\mathfrak{b}$ is the upper-right triangular matrices of $\mathfrak{sl}(r,\mathbb{C})$, and the induced Borel subgroup B is the upper-right triangular matrices of $\mathbf{SL}(r,\mathbb{C})$.

The standard parabolic subalgebras $\mathfrak{p}_I$ are the subalgebras of upper-right triangular block matrices of $\mathfrak{sl}(r,\mathbb{C})$, where the sizes of the blocks are coded by $I \subseteq \triangle$. Similar statements hold for the standard parabolic subgroups P_I of $\mathbf{SL}(r,\mathbb{C})$.

For $i = 1, \ldots, r-1$, if $\alpha_{i,i+1} \in I$, there exists a diagonal block in $\mathfrak{p}_I$ and P_I ending at the i-th row, and another diagonal block starting at the $i+1$-st row.

Note that maximal standard parabolics of $\mathfrak{sl}(r,\mathbb{C})$ and $\mathbf{SL}(r,\mathbb{C})$, i.e., standard parabolics corresponding to singletons $I = \{*\} \subseteq \triangle$, are the standard parabolics that have precisely two diagonal blocks.

We now see how $I \subseteq \triangle$ determines the sizes of blocks in examples. For $r = 4$, we have $I \subseteq \triangle = \{\alpha_{1,2}, \alpha_{2,3}, \alpha_{3,4}\}$, and:

- For $I = \{\alpha_{1,2}\}$, $\mathfrak{p}_I$ and P_I have a lower-right 3×3-block:

$$\mathfrak{p}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}, \sum_{i=1}^{4} a_{ii} = 0 \right\}, \quad P_I = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \in \mathbf{SL}(4,\mathbb{C}) \right\}. \tag{1.2.11}$$

- For $I = \{\alpha_{2,3}, \alpha_{3,4}\}$, $\mathfrak{p}_I$ and P_I have an upper-left 2×2-block:

$$\mathfrak{p}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}, \sum_{i=1}^{4} a_{ii} = 0 \right\}, \quad P_I = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in \mathbf{SL}(4,\mathbb{C}) \right\}. \tag{1.2.12}$$

(Step 4) For the Levi decompositions $\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$ and $P_I \simeq U_I \rtimes L_I$, we have that $\mathfrak{l}_I$ and L_I consist of the diagonal block matrices of $\mathfrak{p}_I$ and P_I, whereas $\mathfrak{u}_I$ consists of the nilpotent matrices of $\mathfrak{p}_I$, and U_I consists of the unipotent matrices of P_I.

Note that for a certain $r_1, \ldots, r_l \in \mathbb{N}$, such that $r_1 + \ldots + r_l = r$, L_I is isomorphic to the subgroup of matrices of $\mathbf{GL}(r_1,\mathbb{C}) \times \ldots \times \mathbf{GL}(r_l,\mathbb{C})$, of determinant 1.

For $r = 4$, we have $I \subseteq \triangle = \{\alpha_{1,2}, \alpha_{2,3}, \alpha_{3,4}\}$, and:

- For $I = \{\alpha_{2,3}, \alpha_{3,4}\} \subseteq \triangle$, the Levi decompositions of $\mathfrak{p}_I$ and P_I appear as:

$$\mathfrak{l}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}, \sum_{i=1}^{4} a_{ii} = 0 \right\}, \quad L_I = \left\{ \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \in \mathbf{SL}(4,\mathbb{C}) \right\}, \tag{1.2.13}$$

$$\mathfrak{u}_I = \left\{ \begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \qquad U_I = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbf{SL}(4,\mathbb{C}) \right\}. \tag{1.2.14}$$

In this case, it is easy to directly verify that $\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$ and $P_I \simeq U_I \rtimes L_I$, and $L_I \simeq P_I/U_I$.

Example 1.2.2. *The connected complex reductive group* $\mathbf{GL}(r,\mathbb{C})$.

From (a) of Example 1.1.7, we know that $\mathfrak{gl}(r,\mathbb{C}) = \mathfrak{gl}(r,\mathbb{C})_{ss} \oplus \mathfrak{z}(\mathfrak{gl}(r,\mathbb{C}))$, where $\mathfrak{gl}(r,\mathbb{C})_{ss} = \mathfrak{sl}(r,\mathbb{C})$ and $\mathfrak{z}(\mathfrak{gl}(r,\mathbb{C})) = \operatorname{span}_{\mathbb{C}}(I_r)$. Through this and (a) of Remark 1.1.9, we can make use of results from Example 1.2.1.

(Step 1) For $r = 1$, $T = \mathbf{GL}(1,\mathbb{C})$ is a Cartan subgroup of $\mathbf{GL}(1,\mathbb{C})$ isomorphic to $\mathbb{C}^\times$, with the Lie algebra $\mathfrak{t} = \mathfrak{gl}(1,\mathbb{C})$, which has rank 1 and semisimple rank 0.

For $r \geq 2$, let $\mathfrak{h}$ be the Cartan subalgebra of diagonal matrices of $\mathfrak{sl}(r,\mathbb{C})$, then due to (a) of Remark 1.1.9, $\mathfrak{t} = \mathfrak{h} \oplus \operatorname{span}_{\mathbb{C}}(I_r)$ is a Cartan subalgebra of $\mathfrak{gl}(r,\mathbb{C})$ of diagonal matrices, which has rank r and semisimple rank $r-1$.

This is the Lie algebra of the Cartan subgroup T of $\mathbf{GL}(r,\mathbb{C})$ of diagonal matrices, isomorphic to $(\mathbb{C}^\times)^r$.

(Step 2) For $r = 1$, the root space decomposition is $\mathfrak{t} = \mathfrak{gl}(1,\mathbb{C})$, with no roots.

For $r \geq 2$, due to (d) of Remark 1.1.11, there exists a bijection between $\Phi(\mathfrak{gl}(r,\mathbb{C}),\mathfrak{t})$ and $\Phi(\mathfrak{sl}(r,\mathbb{C}),\mathfrak{h})$ preserving weight spaces. Thus, using notation from (Step 2) of Example 1.2.1, the roots are:

$$\Phi(\mathfrak{gl}(r,\mathbb{C}),\mathfrak{t}) = \{\alpha_{ij} | i,j = 1,\ldots,r : i \neq j\}, \tag{1.2.15}$$

with the weight spaces $\mathfrak{gl}(r,\mathbb{C})_{\alpha_{ij}} = \operatorname{span}_{\mathbb{C}}(E_{ij})$, $i,j = 1,\ldots,r$, $i \neq j$, such that the root space decomposition is:

$$\mathfrak{gl}(r,\mathbb{C}) = \mathfrak{t} \oplus \bigoplus_{\alpha_{ij} \in \Phi(\mathfrak{gl}(r,\mathbb{C}),\mathfrak{t})} \operatorname{span}_{\mathbb{C}}(E_{ij}). \tag{1.2.16}$$

Similarly to Example 1.2.1, we choose the positive roots $\Phi(\mathfrak{gl}(r,\mathbb{C}),\mathfrak{t})^+ = \{\alpha_{ij} | i,j = 1,\ldots,r : i < j\}$, inducing the simple roots $\triangle = \{\alpha_{i,i+1} | i = 1,\ldots,r-1\}$.

In the sense of the Dynkin classification from [Hum72, III Theorem 11.4], $\Phi(\mathfrak{gl}(r,\mathbb{C}),\mathfrak{t})$ is a root system of type A_{r-1}.

For the rest of this example, we assume $r \geq 2$.

(Step 3) The induced Borel subalgebra $\mathfrak{b}$ is the upper-right triangular matrices of $\mathfrak{gl}(r,\mathbb{C})$, and the induced Borel subgroup B is the upper-right triangular matrices of $\mathbf{GL}(r,\mathbb{C})$.

The standard parabolic subalgebras $\mathfrak{p}_I$ are the subalgebras of upper-right triangular block matrices of $\mathfrak{gl}(r,\mathbb{C})$, of which the sizes of blocks are coded by $I \subseteq \triangle$, completely analogously to (Step 3) of Example 1.2.1. Similar statements hold for the standard parabolic subgroups P_I of $\mathbf{GL}(r,\mathbb{C})$.

The only difference to Example 1.2.1 is that the subalgebras $\mathfrak{p}_I$ are not trace-free, and the subgroups P_I do not have determinant 1.

(Step 4) The Levi decompositions $\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$ and $P_I \simeq U_I \rtimes L_I$ are analogous to (Step 4) of Example 1.2.1.

The only difference to Example 1.2.1 is that for a certain $r_1,\ldots,r_l \in \mathbb{N}$, such that $r_1 + \ldots + r_l = r$, L_I is isomorphic to $\mathbf{GL}(r_1,\mathbb{C}) \times \ldots \times \mathbf{GL}(r_l,\mathbb{C})$, and not a proper subgroup.

1.2.2. The case of $\mathbf{SO}(r,\mathbb{C})$

We now investigate $\mathbf{SO}(r,\mathbb{C})$, $r \geq 2$, handling the odd cases $r = 2n+1$ and the even cases $r = 2n$ separately, as their root systems are not of the same type.

Since root space decompositions, Cartan, Borel and parabolic subgroups, are invariant under isomorphisms of complex algebraic groups, we can investigate complex reductive groups isomorphic to $\mathbf{SO}(r,\mathbb{C})$, where these constructions are most easily described.

Matrices in $\mathbf{SO}(r,\mathbb{C})$ are those in $\mathbf{GL}(r,\mathbb{C})$ preserving the standard nondegenerate quadratic form $Q(x) = x_1^2 + \ldots + x_r^2$ on $\mathbb{C}^r$. Since all quadratic forms on $\mathbb{C}^r$ are equivalent, we can choose another quadratic form $\overline{Q}(x) = x_1 x_r + \ldots + x_r x_1$ on $\mathbb{C}^r$ to induce another complex reductive group $\overline{\mathbf{SO}}(r,\mathbb{C})$ of matrices in $\mathbf{GL}(r,\mathbb{C})$ preserving $\overline{Q}$. This group is isomorphic to $\mathbf{SO}(r,\mathbb{C})$, with the Lie algebra $\overline{\mathfrak{so}}(r,\mathbb{C})$. Explicitly, we have:

$$K_r = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathbf{GL}(r,\mathbb{C}), \tag{1.2.17}$$

$$\overline{\mathbf{SO}}(r,\mathbb{C}) = \{A \in \mathbf{SL}(r,\mathbb{C}) | A^T K_r A = K_r\}, \tag{1.2.18}$$

$$\overline{\mathfrak{so}}(r,\mathbb{C}) = \{X \in \mathfrak{gl}(r,\mathbb{C}) | K_r X + X^T K_r = 0\}. \tag{1.2.19}$$

Analogously, we have:

$$\overline{\mathbf{O}}(r,\mathbb{C}) = \{A \in \mathbf{GL}(r,\mathbb{C}) | A^T K_r A = K_r\}, \tag{1.2.20}$$

$$\overline{\mathfrak{o}}(r,\mathbb{C}) = \overline{\mathfrak{so}}(r,\mathbb{C}). \tag{1.2.21}$$

which are isomorphic to $\mathbf{O}(r,\mathbb{C})$ and $\mathfrak{o}(r,\mathbb{C})$.

The advantage of $\overline{\mathbf{SO}}(r,\mathbb{C})$ over $\mathbf{SO}(r,\mathbb{C})$ is that in the odd $r = 2n+1$ case, certain Cartan, Borel and standard parabolic subgroups of $\mathbf{SL}(r,\mathbb{C})$ can be intersected with $\overline{\mathbf{SO}}(r,\mathbb{C})$ to obtain Cartan, Borel and standard parabolic subgroups of $\overline{\mathbf{SO}}(r,\mathbb{C})$, as seen in [MT12, Example 6.7]. This cannot be done for $\mathbf{SO}(r,\mathbb{C})$, for which the approach in [Kna88, II.1 Example 2, Example 4] can be taken instead.

EXAMPLE 1.2.3. *The connected complex semisimple group* $\overline{\mathbf{SO}}(2n+1,\mathbb{C})$.
(STEP 1) For $z_1, \ldots, z_n \in \mathbb{C}$ and $s_1, \ldots, s_n \in \mathbb{C}^\times$, we define:

$$Z_{z_1,\ldots,z_n} = \begin{pmatrix} z_1 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & z_n & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -z_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & -z_1 \end{pmatrix} \in \overline{\mathfrak{so}}(2n+1,\mathbb{C}), \tag{1.2.22}$$

$$S_{s_1,\ldots,s_n} = \begin{pmatrix} s_1 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & s_n & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & s_n^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & s_1^{-1} \end{pmatrix} \in \overline{\mathbf{SO}}(2n+1,\mathbb{C}). \tag{1.2.23}$$

We claim that the subgroup $\overline{T} = \{S_{s_1,\ldots,s_n} | s_1, \ldots, s_n \in \mathbb{C}^\times\}$ of diagonal matrices is a Cartan subgroup of $\overline{\mathbf{SO}}(2n+1,\mathbb{C})$, implying that its Lie algebra $\overline{\mathfrak{t}} = \{Z_{z_1,\ldots,z_n} | z_1, \ldots, z_n \in \mathbb{C}\}$ is a Cartan subalgebra of $\overline{\mathfrak{so}}(2n+1,\mathbb{C})$, which therefore has rank n, equal to its semisimple rank.

Since $\overline{T}$ is a torus of $\overline{\mathbf{SO}}(2n+1,\mathbb{C})$ isomorphic to $(\mathbb{C}^\times)^n$, it is contained within a maximal torus $\overline{M}$ of $\overline{\mathbf{SO}}(2n+1,\mathbb{C})$, for which we claim that $\overline{M}=\overline{T}$. Choose $s_1,\dots,s_n \in \mathbb{C}^\times$ such that $s_1,\dots,s_n,s_1^{-1},\dots,s_n^{-1}$ are all distinct. Using centralizer subgroups of $\overline{\mathbf{SO}}(2n+1,\mathbb{C})$ and $\mathbf{GL}(2n+1,\mathbb{C})$, we have:

$$\overline{M} \subseteq \mathbf{Z}_{\overline{\mathbf{SO}}(2n+1,\mathbb{C})}(\overline{T}) \subseteq \mathbf{Z}_{\mathbf{GL}(2n+1,\mathbb{C})}(S_{s_1,\dots,s_n}), \tag{1.2.24}$$

where $\mathbf{Z}_{\mathbf{GL}(2n+1,\mathbb{C})}(S_{s_1,\dots,s_n})$ consists only of diagonal matrices. Thus, we have $\overline{M}=\overline{T}$.

Note that these Cartans are the Cartans of $\mathfrak{sl}(2n+1,\mathbb{C})$ and $\mathbf{SL}(2n+1,\mathbb{C})$ from (Step 1) of Example 1.2.1, intersected with $\overline{\mathfrak{so}}(2n+1,\mathbb{C})$ and $\overline{\mathbf{SO}}(2n+1,\mathbb{C})$.

(Step 2) We use notation from Example 1.2.1. For $i,j=1,\dots,n$, $i\neq j$, since $e_i=-e_{2n+2-i}$, we can use the calculations in (1.2.1)...(1.2.8) to follow that:

$$\overline{\mathfrak{so}}(2n+1,\mathbb{C})_\alpha = \operatorname{span}_{\mathbb{C}}(E_{ij}-E_{2n+2-j,2n+2-i}), \qquad \alpha=e_i-e_j, \tag{1.2.25}$$
$$\overline{\mathfrak{so}}(2n+1,\mathbb{C})_\alpha = \operatorname{span}_{\mathbb{C}}(E_{i,2n+2-j}-E_{j,2n+2-i}), \qquad \alpha=e_i+e_j, \tag{1.2.26}$$
$$\overline{\mathfrak{so}}(2n+1,\mathbb{C})_\alpha = \operatorname{span}_{\mathbb{C}}(E_{2n+2-i,j}-E_{2n+2-j,i}), \qquad \alpha=-e_i-e_j, \tag{1.2.27}$$
$$\overline{\mathfrak{so}}(2n+1,\mathbb{C})_\alpha = \operatorname{span}_{\mathbb{C}}(E_{i,n+1}-E_{n+1,2n+2-i}), \qquad \alpha=e_i, \tag{1.2.28}$$
$$\overline{\mathfrak{so}}(2n+1,\mathbb{C})_\alpha = \operatorname{span}_{\mathbb{C}}(E_{2n+2-i,n+1}-E_{n+1,i}), \qquad \alpha=-e_i. \tag{1.2.29}$$

Since these span $\overline{\mathfrak{so}}(2n+1,\mathbb{C})$, we found the root space decomposition, with the roots:

$$\Phi(\overline{\mathfrak{so}}(2n+1,\mathbb{C}),\bar{\mathfrak{t}}) = \{e_i-e_j, e_i+e_j, -e_i-e_j, e_k, -e_k | i,j,k=1,\dots,n : i\neq j\}. \tag{1.2.30}$$

The following set:

$$\Phi(\overline{\mathfrak{so}}(2n+1,\mathbb{C}),\bar{\mathfrak{t}})^+ = \{e_i-e_j, e_i+e_j, e_k | i,j,k=1,\dots,n : i<j\}, \tag{1.2.31}$$

forms positive roots, inducing the simple roots $\triangle=\{e_i-e_{i+1}, e_n | i=1,\dots,n-1\}$.

In the sense of the Dynkin classification from [Hum72, III Theorem 11.4], $\Phi(\overline{\mathfrak{so}}(2n+1,\mathbb{C}),\bar{\mathfrak{t}})$ is a root system of type B_n.

(Step 3) The induced Borel subalgebra $\bar{\mathfrak{b}}$ is the upper-right triangular matrices of $\overline{\mathfrak{so}}(2n+1,\mathbb{C})$, and the induced Borel subgroup $\overline{B}$ is the upper-right triangular matrices of $\overline{\mathbf{SO}}(2n+1,\mathbb{C})$.

The standard parabolic subalgebras $\bar{\mathfrak{p}}_I$ are the subalgebras of upper-right triangular block matrices of $\overline{\mathfrak{so}}(2n+1,\mathbb{C})$, of which the sizes of blocks are coded by $I\subseteq\triangle$. Similar statements hold for the standard parabolic subgroups $\overline{P}_I$ of $\overline{\mathbf{SO}}(2n+1,\mathbb{C})$.

For $i=1,\dots,n$, if $e_i-e_{i+1}\in I$ or $e_i\in I$, there exists diagonal blocks in $\bar{\mathfrak{p}}_I$ and $\overline{P}_I$ ending at the i-th row and at the $2n+1-i$-th row, and diagonal blocks starting at the $i+1$-st row and at the $2n+2-i$-th row.

Note that these Borels and standard parabolics are the Borels and standard parabolics of $\mathfrak{sl}(2n+1,\mathbb{C})$ and $\mathbf{SL}(2n+1,\mathbb{C})$, from (Step 3) of Example 1.2.1, intersected with $\overline{\mathfrak{so}}(2n+1,\mathbb{C})$ and $\overline{\mathbf{SO}}(2n+1,\mathbb{C})$.

We now see how $I\subseteq\triangle$ determines the sizes of blocks in examples. For $n=2$, we have $I\subseteq\triangle=\{e_1-e_2,e_2\}$, and:

- For $I=\{e_1-e_2\}$, $\bar{\mathfrak{p}}_I$ and $\overline{P}_I$ have a middle 3×3-block:

$$\bar{\mathfrak{p}}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ 0 & a_{22} & a_{23} & 0 & -a_{14} \\ 0 & a_{32} & 0 & -a_{23} & -a_{13} \\ 0 & 0 & -a_{32} & -a_{22} & -a_{12} \\ 0 & 0 & 0 & 0 & -a_{11} \end{pmatrix} \right\}, \quad \overline{P}_I = \left\{ \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \in \overline{\mathbf{SO}}(5,\mathbb{C}) \right\}. \tag{1.2.32}$$

• For $I = \{e_2\}$, $\overline{\mathfrak{p}}_I$ and $\overline{P}_I$ have upper-left and lower-right 2×2-blocks:

$$\overline{\mathfrak{p}}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & 0 & -a_{14} \\ 0 & 0 & 0 & -a_{23} & -a_{13} \\ 0 & 0 & 0 & -a_{22} & -a_{12} \\ 0 & 0 & 0 & -a_{21} & -a_{11} \end{pmatrix} \right\}, \quad \overline{P}_I = \left\{ \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \in \overline{\mathbf{SO}}(5, \mathbb{C}) \right\}. \tag{1.2.33}$$

(STEP 4) For the Levi decompositions $\overline{\mathfrak{p}}_I = \overline{\mathfrak{l}}_I \oplus \overline{\mathfrak{u}}_I$ and $\overline{P}_I \simeq \overline{U}_I \rtimes \overline{L}_I$, we have that $\overline{\mathfrak{l}}_I$ and $\overline{L}_I$ consist of the diagonal block matrices of $\overline{\mathfrak{p}}_I$ and $\overline{P}_I$, whereas $\overline{\mathfrak{u}}_I$ consists of the nilpotent matrices of $\overline{\mathfrak{p}}_I$, and $\overline{U}_I$ consists of the unipotent matrices of $\overline{P}_I$.

Note that for a certain $r_1, \dots, r_{l-1} \in \mathbb{N}, n_l \in \mathbb{N}_0$, such that $r_1 + \ldots + r_{l-1} \le n$ and $n + 1 \le r_1 + \ldots + r_{l-1} + 2n_l + 1 \le 2n + 1$, $\overline{L}_I$ is isomorphic to the group:

$$\mathbf{GL}(r_1, \mathbb{C}) \times \ldots \times \mathbf{GL}(r_{l-1}, \mathbb{C}) \times \mathbf{SO}(2n_l + 1, \mathbb{C}). \tag{1.2.34}$$

For $n = 2$, we have $I \subseteq \triangle = \{e_1 - e_2, e_2\}$, and:

• For $I = \{e_2\}$, the Levi decompositions of $\overline{\mathfrak{p}}_I$ and $\overline{P}_I$ appear as:

$$\overline{\mathfrak{l}}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_{22} & -a_{12} \\ 0 & 0 & 0 & -a_{21} & -a_{11} \end{pmatrix} \right\}, \quad \overline{L}_I = \left\{ \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \in \overline{\mathbf{SO}}(5, \mathbb{C}) \right\}, \tag{1.2.35}$$

$$\overline{\mathfrak{u}}_I = \left\{ \begin{pmatrix} 0 & 0 & a_{13} & a_{14} & 0 \\ 0 & 0 & a_{23} & 0 & -a_{14} \\ 0 & 0 & 0 & -a_{23} & -a_{13} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \quad \overline{U}_I = \left\{ \begin{pmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \overline{\mathbf{SO}}(5, \mathbb{C}) \right\}. \tag{1.2.36}$$

In this case, it is easy to directly verify that $\overline{\mathfrak{p}}_I = \overline{\mathfrak{l}}_I \oplus \overline{\mathfrak{u}}_I$ and $\overline{P}_I \simeq \overline{U}_I \rtimes \overline{L}_I$, and $\overline{L}_I \simeq \overline{P}_I / \overline{U}_I$.

EXAMPLE 1.2.4. *The connected complex reductive group* $\overline{\mathbf{SO}}(2, \mathbb{C})$*, and the connected complex semisimple group* $\overline{\mathbf{SO}}(2n, \mathbb{C})$*,* $n \ge 2$*.*

(STEP 1) For $z_1, \dots, z_n \in \mathbb{C}$ and $s_1, \dots, s_n \in \mathbb{C}^\times$, we define:

$$Z_{z_1,\dots,z_n} = \begin{pmatrix} z_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & z_n & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -z_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 & -z_1 \end{pmatrix} \in \overline{\mathfrak{so}}(2n, \mathbb{C}), \tag{1.2.37}$$

$$S_{s_1,\dots,s_n} = \begin{pmatrix} s_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & s_n & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & s_n^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 & s_1^{-1} \end{pmatrix} \in \overline{\mathbf{SO}}(2n, \mathbb{C}). \tag{1.2.38}$$

Using the same argument from (Step 1) of Example 1.2.3, the subgroup $\overline{T} = \{S_{s_1,\ldots,s_n} | s_1, \ldots, s_n \in \mathbb{C}^\times\}$ of diagonal matrices is a Cartan subgroup of $\overline{\mathbf{SO}}(2n,\mathbb{C})$, implying that its Lie algebra $\bar{\mathfrak{t}} = \{Z_{z_1,\ldots,z_n} | z_1, \ldots, z_n \in \mathbb{C}\}$ is a Cartan subalgebra of $\overline{\mathfrak{so}}(2n,\mathbb{C})$, which therefore has rank n, equal to its semisimple rank. In particular for $n = 1$, we have $\bar{\mathfrak{t}} = \overline{\mathfrak{so}}(2,\mathbb{C})$ and $\overline{T} = \overline{\mathbf{SO}}(2,\mathbb{C})$.

Note that these Cartans are the Cartans of $\mathfrak{sl}(2n,\mathbb{C})$ and $\mathbf{SL}(2n,\mathbb{C})$ from (Step 1) of Example 1.2.1, intersected with $\overline{\mathfrak{so}}(2n,\mathbb{C})$ and $\overline{\mathbf{SO}}(2n,\mathbb{C})$.

(Step 2) For $n = 1$, the root space decomposition is $\mathfrak{t} = \overline{\mathfrak{so}}(2,\mathbb{C})$, with no roots.

For $n \geq 2$, we use notation from Example 1.2.1. For $i, j = 1, \ldots, n$, $i \neq j$, since $e_i = -e_{2n+1-i}$, we can use the calculations in (1.2.1)…(1.2.8) to follow that:

$$\overline{\mathfrak{so}}(2n,\mathbb{C})_\alpha = \mathrm{span}_{\mathbb{C}}(E_{ij} - E_{2n+1-j,2n+1-i}), \qquad \alpha = e_i - e_j, \tag{1.2.39}$$
$$\overline{\mathfrak{so}}(2n,\mathbb{C})_\alpha = \mathrm{span}_{\mathbb{C}}(E_{i,2n+1-j} - E_{j,2n+1-i}), \qquad \alpha = e_i + e_j, \tag{1.2.40}$$
$$\overline{\mathfrak{so}}(2n,\mathbb{C})_\alpha = \mathrm{span}_{\mathbb{C}}(E_{2n+1-i,j} - E_{2n+1-j,i}), \qquad \alpha = -e_i - e_j. \tag{1.2.41}$$

Since these span $\overline{\mathfrak{so}}(2n,\mathbb{C})$, we found the root space decomposition, with the roots:

$$\Phi(\overline{\mathfrak{so}}(2n,\mathbb{C}),\bar{\mathfrak{t}}) = \{e_i - e_j, e_i + e_j, -e_i - e_j | i, j = 1, \ldots, n : i \neq j\}. \tag{1.2.42}$$

The following set:

$$\Phi(\overline{\mathfrak{so}}(2n,\mathbb{C}),\bar{\mathfrak{t}})^+ = \{e_i - e_j, e_i + e_j | i, j = 1, \ldots, n : i < j\}, \tag{1.2.43}$$

forms positive roots, inducing the simple roots $\triangle = \{e_i - e_{i+1}, e_{n-1} + e_n | i = 1, \ldots, n-1\}$.

In the sense of the Dynkin classification from [Hum72, III Theorem 11.4], $\Phi(\overline{\mathfrak{so}}(2n,\mathbb{C}),\bar{\mathfrak{t}})$ is a root system of type D_n.

For the rest of this example, we assume $n \geq 2$.

(Step 3) The induced Borel subalgebra $\bar{\mathfrak{b}}$ is the upper-right triangular matrices of $\overline{\mathfrak{so}}(2n,\mathbb{C})$, and the induced Borel subgroup $\overline{B}$ is the upper-right triangular matrices of $\overline{\mathbf{SO}}(2n,\mathbb{C})$.

Note that these Borels are the Borels of $\mathfrak{sl}(2n,\mathbb{C})$ and $\mathbf{SL}(2n,\mathbb{C})$, from (Step 3) of Example 1.2.1, intersected with $\overline{\mathfrak{so}}(2n,\mathbb{C})$ and $\overline{\mathbf{SO}}(2n,\mathbb{C})$.

The standard parabolic subalgebras $\bar{\mathfrak{p}}_I$ are the subalgebras of $\overline{\mathfrak{so}}(2n,\mathbb{C})$ containing $\bar{\mathfrak{b}}$. Isof $e_{n-1} - e_n \notin I$ or $e_{n-1} + e_n \in I$, then $\bar{\mathfrak{p}}_I$ consists of upper-right triangular block matrices of $\overline{\mathfrak{so}}(2n,\mathbb{C})$. If $e_{n-1} - e_n \in I$ and $e_{n-1} + e_n \notin I$, $\bar{\mathfrak{p}}_I$ does not obtain this form, but is nevertheless isomorphic to another standard parabolic subalgebra $\bar{\mathfrak{p}}_J$ of upper-right triangular block matrices of $\overline{\mathfrak{so}}(2n,\mathbb{C})$. This isomorphism is found by permuting the columns and rows of matrices in $\overline{\mathfrak{so}}(2n,\mathbb{C})$. Similar statements hold for the standard parabolic subgroups $\overline{P}_I$ of $\overline{\mathbf{SO}}(2n,\mathbb{C})$.

Given $e_{n-1} - e_n \notin I$ or $e_{n-1} + e_n \in I$, for $i = 1, \ldots, n-1$, if $e_i - e_{i+1} \in I$, there exists diagonal blocks in $\bar{\mathfrak{p}}_I$ and $\overline{P}_I$ ending at the i-th row and at the $2n-i$-th row, and diagonal blocks starting at the $i+1$-st row and at the $2n+1-i$-th row. If $e_{n-1} + e_n \in I$, there exists diagonal blocks ending at the n-th row and starting at the $n+1$-st row.

We now see how $I \subseteq \triangle$ determines the sizes of blocks in examples. For $n = 2$, we have $I \subseteq \triangle = \{e_1 - e_2, e_1 + e_2\}$, and:

- For $I = \{e_1 - e_2\}$, $\bar{\mathfrak{p}}_I$ and $\overline{P}_I$ are not in the form of upper-right triangular block matrices:

$$\bar{\mathfrak{p}}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & 0 & -a_{13} \\ a_{31} & 0 & -a_{22} & -a_{12} \\ 0 & -a_{31} & 0 & -a_{11} \end{pmatrix} \right\}, \quad \overline{P}_I = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{pmatrix} \in \overline{\mathbf{SO}}(4,\mathbb{C}) \right\}. \tag{1.2.44}$$

- For $I = \{e_1 + e_2\}$, $\overline{\mathfrak{p}}_I$ and $\overline{P}_I$ have upper-left and lower-right 2×2-blocks:

$$\overline{\mathfrak{p}}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & 0 & -a_{13} \\ 0 & 0 & -a_{22} & -a_{12} \\ 0 & 0 & -a_{21} & -a_{11} \end{pmatrix} \right\}, \quad \overline{P}_I = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in \overline{\mathbf{SO}}(4, \mathbb{C}) \right\}. \quad (1.2.45)$$

Note that these standard parabolics are isomorphic to each other, through permutations of the middle two rows and columns.

(STEP 4) Given $e_{n-1}-e_n \notin I$ or $e_{n-1}+e_n \in I$, for the Levi decompositions $\overline{\mathfrak{p}}_I = \bar{\mathfrak{l}}_I \oplus \bar{\mathfrak{u}}_I$ and $\overline{P}_I \simeq \overline{U}_I \rtimes \overline{L}_I$, we have that $\bar{\mathfrak{l}}_I$ and $\overline{L}_I$ consist of the diagonal block matrices of $\overline{\mathfrak{p}}_I$ and $\overline{P}_I$, whereas $\bar{\mathfrak{u}}_I$ consists of the nilpotent matrices of $\overline{\mathfrak{p}}_I$, and $\overline{U}_I$ consists of the unipotent matrices of $\overline{P}_I$.

Note that for all $I \subseteq \triangle$, for a certain $r_1, \dots, r_{l-1} \in \mathbb{N}, n_l \in \mathbb{N}_0$, such that $r_1 + \dots + r_{l-1} \leq n$ and $n \leq r_1 + \dots + r_{l-1} + 2n_l \leq 2n$, $\overline{L}_I$ is isomorphic to the group:

$$\mathbf{GL}(r_1, \mathbb{C}) \times \dots \times \mathbf{GL}(r_{l-1}, \mathbb{C}) \times \mathbf{SO}(2n_l, \mathbb{C}). \quad (1.2.46)$$

For $n = 2$, we have $I \subseteq \triangle = \{e_1 - e_2, e_1 + e_2\}$, and:

- For $I = \{e_1 + e_2\}$, the Levi decompositions of $\overline{\mathfrak{p}}_I$ and $\overline{P}_I$ appear as:

$$\bar{\mathfrak{l}}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & -a_{22} & -a_{12} \\ 0 & 0 & -a_{21} & -a_{11} \end{pmatrix} \right\}, \quad \overline{L}_I = \left\{ \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in \overline{\mathbf{SO}}(4, \mathbb{C}) \right\}, \quad (1.2.47)$$

$$\bar{\mathfrak{u}}_I = \left\{ \begin{pmatrix} 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & -a_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \quad \overline{U}_I = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \overline{\mathbf{SO}}(4, \mathbb{C}) \right\}. \quad (1.2.48)$$

In this case, it is easy to directly verify that $\overline{\mathfrak{p}}_I = \bar{\mathfrak{l}}_I \oplus \bar{\mathfrak{u}}_I$ and $\overline{P}_I \simeq \overline{U}_I \rtimes \overline{L}_I$, and $\overline{L}_I \simeq \overline{P}_I / \overline{U}_I$.

Example 1.2.3 and Example 1.2.4 both apply analogously to $\overline{\mathbf{O}}(r, \mathbb{C})$, where the identity components of parabolics of $\overline{\mathbf{O}}(r, \mathbb{C})$ are parabolics of $\overline{\mathbf{SO}}(r, \mathbb{C})$.

Through a fixed isomorphism $\mathbf{SO}(r, \mathbb{C}) \simeq \overline{\mathbf{SO}}(r, \mathbb{C})$, these examples induce analogous constructions on $\mathbf{SO}(r, \mathbb{C})$, such as parabolic subgroups P_I of $\mathbf{SO}(r, \mathbb{C})$, corresponding to $I \subseteq \triangle$.

1.2.3. The case of $\mathbf{Sp}(2n, \mathbb{C})$

We finally investigate $\mathbf{Sp}(2n, \mathbb{C})$. Similarly to the orthogonal case, by altering the preserved symplectic form, we can induce another complex algebraic group $\overline{\mathbf{Sp}}(2n, \mathbb{C})$ isomorphic to $\mathbf{Sp}(2n, \mathbb{C})$, with the Lie algebra $\overline{\mathfrak{sp}}(2n, \mathbb{C})$, given by:

$$M_{2n} = \begin{pmatrix} 0 & K_n \\ -K_n & 0 \end{pmatrix} \in \mathbf{GL}(2n, \mathbb{C}), \quad (1.2.49)$$

$$\overline{\mathbf{Sp}}(2n, \mathbb{C}) = \{A \in \mathbf{GL}(2n, \mathbb{C}) | A^T M_{2n} A = M_{2n}\}, \quad (1.2.50)$$

$$\overline{\mathfrak{sp}}(2n, \mathbb{C}) = \{X \in \mathfrak{gl}(2n, \mathbb{C}) | M_{2n} X + X^T M_{2n} = 0\}, \quad (1.2.51)$$

such that certain Cartans, Borels and standard parabolics of $\mathbf{SL}(2n, \mathbb{C})$ and can be intersected with $\overline{\mathbf{Sp}}(2n, \mathbb{C})$ to obtain Cartans, Borels and standard parabolics of $\overline{\mathbf{Sp}}(2n, \mathbb{C})$.

EXAMPLE 1.2.5. *The connected complex semisimple group* $\overline{\mathbf{Sp}}(2n, \mathbb{C})$.

(STEP 1) For $z_1, \dots, z_n \in \mathbb{C}$ and $s_1, \dots, s_n \in \mathbb{C}^\times$, we define $Z_{z_1,\dots,z_n} \in \overline{\mathfrak{sp}}(2n, \mathbb{C})$ and $S_{s_1,\dots,s_n} \in \overline{\mathbf{Sp}}(2n, \mathbb{C})$ identically to (STEP 1) of Example 1.2.4. Using the same argument from (STEP 1) of Example 1.2.3, the subgroup $\overline{T} = \{S_{s_1,\dots,s_n} | s_1, \dots, s_n \in \mathbb{C}^\times\}$

of diagonal matrices is a Cartan subgroup of $\overline{\mathbf{Sp}}(2n,\mathbb{C})$, implying that its Lie algebra $\bar{\mathfrak{t}} = \{Z_{z_1,\dots,z_n}|z_1,\dots,z_n \in \mathbb{C}\}$ is a Cartan subalgebra of $\overline{\mathfrak{sp}}(2n,\mathbb{C})$, which therefore has rank n, equal to its semisimple rank.

Note that these Cartans are the Cartans of $\mathfrak{sl}(2n,\mathbb{C})$ and $\mathbf{SL}(2n,\mathbb{C})$ from (Step 1) of Example 1.2.1, intersected with $\overline{\mathfrak{sp}}(2n,\mathbb{C})$ and $\overline{\mathbf{Sp}}(2n,\mathbb{C})$.

(Step 2) We use notation from Example 1.2.1. For $i,j = 1,\dots,n$, $i \neq j$, since $e_i = -e_{2n+1-i}$, we can use the calculations in (1.2.1)…(1.2.8) to follow that:

$$\overline{\mathfrak{sp}}(2n,\mathbb{C})_\alpha = \mathrm{span}_{\mathbb{C}}(E_{ij} - E_{2n+1-j,2n+1-i}), \qquad \alpha = e_i - e_j, \tag{1.2.52}$$
$$\overline{\mathfrak{sp}}(2n,\mathbb{C})_\alpha = \mathrm{span}_{\mathbb{C}}(E_{i,2n+1-j} + E_{j,2n+1-i}), \qquad \alpha = e_i + e_j, \tag{1.2.53}$$
$$\overline{\mathfrak{sp}}(2n,\mathbb{C})_\alpha = \mathrm{span}_{\mathbb{C}}(E_{2n+1-i,j} + E_{2n+1-j,i}), \qquad \alpha = -e_i - e_j, \tag{1.2.54}$$
$$\overline{\mathfrak{sp}}(2n,\mathbb{C})_\alpha = \mathrm{span}_{\mathbb{C}}(E_{i,2n+1-i}), \qquad \alpha = 2e_i, \tag{1.2.55}$$
$$\overline{\mathfrak{sp}}(2n,\mathbb{C})_\alpha = \mathrm{span}_{\mathbb{C}}(E_{2n+1-i,i}), \qquad \alpha = -2e_i. \tag{1.2.56}$$

Since these span $\overline{\mathfrak{sp}}(2n,\mathbb{C})$, we found the root space decomposition, with the roots:

$$\Phi(\overline{\mathfrak{sp}}(2n,\mathbb{C}),\bar{\mathfrak{t}}) = \{e_i - e_j, e_i + e_j, -e_i - e_j, 2e_k, -2e_k | i,j,k = 1,\dots,n : i \neq j\}. \tag{1.2.57}$$

The following set:

$$\Phi(\overline{\mathfrak{sp}}(2n,\mathbb{C}),\bar{\mathfrak{t}})^+ = \{e_i - e_j, e_i + e_j, 2e_k | i,j,k = 1,\dots,n : i < j\}, \tag{1.2.58}$$

forms positive roots, inducing the simple roots $\triangle = \{e_i - e_{i+1}, 2e_n | i = 1,\dots,n-1\}$.

In the sense of the Dynkin classification from [Hum72, III Theorem 11.4], $\Phi(\overline{\mathfrak{sp}}(2n,\mathbb{C}),\bar{\mathfrak{t}})$ is a root system of type C_n.

(Step 3) The induced Borel subalgebra $\bar{\mathfrak{b}}$ is the upper-right triangular matrices of $\overline{\mathfrak{sp}}(2n,\mathbb{C})$, and the induced Borel subgroup $\overline{B}$ is the upper-right triangular matrices of $\overline{\mathbf{Sp}}(2n,\mathbb{C})$.

The standard parabolic subalgebras $\bar{\mathfrak{p}}_I$ are the subalgebras of upper-right triangular block matrices of $\overline{\mathfrak{sp}}(2n,\mathbb{C})$, of which the sizes of blocks are coded by $I \subseteq \triangle$. Similar statements hold for the standard parabolic subgroups $\overline{P}_I$ of $\overline{\mathbf{Sp}}(2n,\mathbb{C})$.

For $i = 1,\dots,n$, if $e_i - e_{i+1} \in I$ or $2e_i \in I$, there exists diagonal blocks in $\bar{\mathfrak{p}}_I$ and $\overline{P}_I$ ending at the i-th row and at the $2n-i$-th row, and diagonal blocks starting at the $i+1$-st row and at the $2n+1-i$-th row.

Note that these Borels and standard parabolics are the Borels and standard parabolics of $\mathfrak{sl}(2n,\mathbb{C})$ and $\mathbf{SL}(2n,\mathbb{C})$ from (Step 3) of Example 1.2.1, intersected with $\overline{\mathfrak{sp}}(2n,\mathbb{C})$ and $\overline{\mathbf{Sp}}(2n,\mathbb{C})$.

We now see how $I \subseteq \triangle$ determines the sizes of blocks in examples. For $n = 2$, we have $I \subseteq \triangle = \{e_1 - e_2, 2e_2\}$, and:

- For $I = \{e_1 - e_2\}$, $\bar{\mathfrak{p}}_I$ and $\overline{P}_I$ have a middle 2×2-block:

$$\bar{\mathfrak{p}}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{13} \\ 0 & a_{32} & -a_{22} & -a_{12} \\ 0 & 0 & 0 & -a_{11} \end{pmatrix} \right\}, \quad \overline{P}_I = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in \overline{\mathbf{Sp}}(4,\mathbb{C}) \right\}. \tag{1.2.59}$$

- For $I = \{2e_2\}$, $\bar{\mathfrak{p}}_I$ and $\overline{P}_I$ have upper-left and lower-right 2×2-blocks:

$$\bar{\mathfrak{p}}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{13} \\ 0 & 0 & -a_{22} & -a_{12} \\ 0 & 0 & -a_{21} & -a_{11} \end{pmatrix} \right\}, \quad \overline{P}_I = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in \overline{\mathbf{Sp}}(4,\mathbb{C}) \right\}. \tag{1.2.60}$$

(STEP 4) For the Levi decompositions $\overline{\mathfrak{p}}_I = \overline{\mathfrak{l}}_I \oplus \overline{\mathfrak{u}}_I$ and $\overline{P}_I \simeq \overline{U}_I \rtimes \overline{L}_I$, we have that $\overline{\mathfrak{l}}_I$ and $\overline{L}_I$ consist of the diagonal block matrices of $\overline{\mathfrak{p}}_I$ and $\overline{P}_I$, whereas $\overline{\mathfrak{u}}_I$ consists of the nilpotent matrices of $\overline{\mathfrak{p}}_I$, and $\overline{\mathfrak{u}}_I$ consists of the unipotent matrices of $\overline{P}_I$.

Note that for a certain $r_1, \dots, r_{l-1} \in \mathbb{N}, n_l \in \mathbb{N}_0$, such that $r_1 + \dots + r_{l-1} \leq n$ and $n \leq r_1 + \dots + r_{l-1} + 2n_l \leq 2n$, $\overline{L}_I$ is isomorphic to the group:

$$\mathbf{GL}(r_1, \mathbb{C}) \times \dots \times \mathbf{GL}(r_{l-1}, \mathbb{C}) \times \mathbf{Sp}(2n_l, \mathbb{C}). \tag{1.2.61}$$

For $n = 2$, we have $I \subseteq \triangle = \{e_1 - e_2, 2e_2\}$, and:

- For $I = \{2e_2\}$, the Levi decompositions of $\overline{\mathfrak{p}}_I$ and $\overline{P}_I$ appear as:

$$\overline{\mathfrak{l}}_I = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & -a_{22} & -a_{12} \\ 0 & 0 & -a_{21} & -a_{11} \end{pmatrix} \right\}, \quad \overline{L}_I = \left\{ \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in \overline{\mathbf{Sp}}(4, \mathbb{C}) \right\}, \tag{1.2.62}$$

$$\overline{\mathfrak{u}}_I = \left\{ \begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \quad \overline{U}_I = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \overline{\mathbf{Sp}}(4, \mathbb{C}) \right\}. \tag{1.2.63}$$

In this case, it is easy to directly verify that $\overline{\mathfrak{p}}_I = \overline{\mathfrak{l}}_I \oplus \overline{\mathfrak{u}}_I$ and $\overline{P}_I \simeq \overline{U}_I \rtimes \overline{L}_I$, and $\overline{L}_I \simeq \overline{P}_I / \overline{U}_I$.

Through a fixed isomorphism $\mathbf{Sp}(2n, \mathbb{C}) \simeq \overline{\mathbf{Sp}}(2n, \mathbb{C})$, this example induces analogous constructions on $\mathbf{Sp}(2n, \mathbb{C})$, such as parabolic subgroups P_I of $\mathbf{Sp}(r, \mathbb{C})$, corresponding to $I \subseteq \triangle$.

REMARK 1.2.6. We have classified the standard parabolic subgroups of $G = \mathbf{GL}(r, \mathbb{C}), \mathbf{SL}(r, \mathbb{C}), \mathbf{SO}(r, \mathbb{C}), \mathbf{Sp}(2n, \mathbb{C})$, such that they all appear isomorphic to stabilizers of flags $\mathcal{F}$ of subspaces of $\mathbb{C}^r$:

$$\mathcal{F} : 0 = V_0 \subsetneq \dots \subsetneq V_l = \mathbb{C}^r, \tag{1.2.64}$$

with respect to the canonical representation $\rho : G \to \mathbf{GL}(\mathbb{C}^r)$.

In the orthogonal (respectively symplectic) case, these flags $\mathcal{F}$ can be refined such that we have for all $i = 0, \dots, l$, that $V_{l-i} = V_i^{\perp}$, with respect to the standard nondegenerate symmetric bilinear form on $\mathbb{C}^r$ (respectively the standard symplectic form).

Such flags $\mathcal{F}$ are called *isotropic flags*, since for all $i = 0, \dots, l$, we have that:

(a) The subspace V_i being *isotropic*, i.e., $V_i \subseteq V_i^{\perp}$, is equivalent to $i \leq l - i$.
(b) The subspace V_i being *coisotropic*, i.e., $V_i^{\perp} \subseteq V_i$, is equivalent to $i \geq l - i$.

As a special case, maximal standard parabolics are stabilizers of minimal flags. In the $\mathbf{GL}(r, \mathbb{C})$ and $\mathbf{SL}(r, \mathbb{C})$ case, they stabilize:

$$\mathcal{F} : 0 = V_0 \subsetneq V_1 \subsetneq V_2 = \mathbb{C}^r. \tag{1.2.65}$$

In the orthogonal and symplectic case, they stabilize:

$$\mathcal{F} : 0 = V_0 \subsetneq V_1 \subseteq V_1^{\perp} \subsetneq V_0^{\perp} = \mathbb{C}^r, \tag{1.2.66}$$

where it is possible that V_1 is both isotropic and coisotropic, i.e., V_1 is *Lagrangian* and $V_1 = V_1^{\perp}$.

These flags turn out to be crucial for many proofs in Chapter 2.

These statements are also proven in [MT12, Proposition 12.13].

CHAPTER 2

Stability in the sense of Ramanathan

In this chapter, we construct holomorphic vector bundles and principal bundles. We define notions of slope-(semi)-stability for vector bundles, from Mumford in [Mum62], and Ramanathan-(semi)-stability for principal bundles, from Ramanathan in [Ram75].

For the structure groups $\mathbf{GL}(r,\mathbb{C})$, and for $r \geq 2$, also $\mathbf{SL}(r,\mathbb{C})$, $\mathbf{SO}(r,\mathbb{C})$ and $\mathbf{Sp}(2n,\mathbb{C})$, we use the parabolic subgroups we described in Chapter 1 to compare these notions of stability.

2.1. Vector bundles and principal bundles

2.1.1. Vector bundles and principal bundles

We introduce holomorphic vector bundles and principal bundles as special types of fiber bundles, where the fibers obtain extra structure. For vector bundles, our main sources are [GH94] and [Ham17], and for principal bundles, we use [Ham17].

Definition 2.1.1. Let $p : \xi \to M$ be a holomorphic map between complex analytic manifolds. The map p is a *fiber bundle* on the *base space* M if:

(i) For all $m \in M$, the set $\xi_m = p^{-1}(m)$ is a complex analytic submanifold of ξ. We call ξ_m the *fiber* of ξ at m.
(ii) For all $m \in M$, there exists an open neighborhood U of m in M, and a biholomorphism $\Phi_m : p^{-1}(U) \to U \times \xi_m$, where for the projection $\mathrm{pr}_1 : U \times \xi_m \to U$, we have $p|_{p^{-1}(U)} = \mathrm{pr}_1 \circ \Phi_m$. This is called the *local triviality* of ξ, and the isomorphisms Φ_m, $m \in M$, are called *local trivializations*.

Usually the map $p : \xi \to M$ is left unlabeled, and we refer to the fiber bundle by ξ.

Unless otherwise stated, all fiber bundles are constructed on the base space M.

We also have *morphisms* of fiber bundles, which are holomorphic maps $\varphi : \xi \to \xi'$ that preserve fibers, similarly to [Ham17, Definition 4.1.6].

Definition 2.1.2. Let E be a fiber bundle, then E is a *vector bundle* of *rank* r if:

(i) For all $m \in M$, the fiber E_m of E at m is a complex vector space of dimension r.
(ii) For all $m \in M$, there exists a local trivialization $\Phi_m : p^{-1}(U) \to U \times E_m$, where for all $u \in U$, $\Phi_m|_{E_u} : E_u \to \{u\} \times E_m$ is an isomorphism of complex vector spaces.

We also have *morphisms* of vector bundles, which are morphisms of fiber bundles that restrict to linear maps on the fibers, similarly to [Ham17, Definition 4.5.7].

Remark 2.1.3. The category of holomorphic vector bundles VecBun_M is not an abelian category, despite having kernels and cokernels. For a morphism $\varphi : E \to F$ of vector bundles, the homomorphism theorem $E/\ker(\varphi) \simeq \mathrm{im}(\varphi)$ may not hold. The reason is that, perhaps counterintuitively, kernels $\ker(\varphi)$ and images $\mathrm{im}(\varphi)$ in VecBun_M do not necessarily coincide with naive fiber-wise kernels and images:

$$\text{“}\ker\text{”}(\varphi) = \bigsqcup_{m\in M} \ker(\varphi|_{E_m}), \quad \text{“im”}(\varphi) = \bigsqcup_{m\in M} \mathrm{im}(\varphi|_{E_m}), \tag{2.1.1}$$

which may not even be vector bundles.

Kernels and images can still be constructed such that $\ker(\varphi) \subseteq$ "$\ker$"(φ) and "im"$(\varphi) \subseteq \operatorname{im}(\varphi)$ as sets, and if $\varphi : E \to F$ is of constant rank, these sets are equal.

Let G be a complex Lie group with neutral element $e \in G$.

DEFINITION 2.1.4. Let ξ be a fiber bundle, then ξ is a *principal-G-bundle* with *structure group G* if:

(i) For all $m \in M$, the fiber ξ_m of ξ at m is equipped with a right-G-torsor structure, i.e., a free and transitive holomorphic right-G-action.
(ii) For all $m \in M$, there exists a local trivialization $\Phi_m : p^{-1}(U) \to U \times \xi_m$, where for all $u \in U$, $\Phi_m|_{\xi_u} : \xi_u \to \{u\} \times \xi_m$ is a G-equivariant biholomorphism.

We also have *morphisms* of principal bundles, which are morphisms of fiber bundles that are G-equivariant on the fibers, similarly to [Ham17, Definition 4.2.17].

REMARK 2.1.5. Due to the G-equivariance of morphisms of principal bundles, morphisms in the category of principal-G-bundles $\mathit{PrincBun}_{G,M}$ are in fact isomorphisms.

Examples of vector bundles include tangent bundles of curves and manifolds, and examples of principal bundles include frame bundles of vector bundles.

We now introduce cocycles, which are a useful tool for characterizing fiber bundles, vector bundles and principal bundles up to isomorphism. Let F be a complex analytic manifold. We denote the group of biholomorphisms from F to itself by $\mathrm{Aut}_{\mathrm{Hol}}(F)$.

DEFINITION 2.1.6. Let $(U_i)_{i\in I}$ be an open covering of M. A family $\sigma_{ij} : U_i \cap U_j \to \mathrm{Aut}_{\mathrm{Hol}}(F)$ of maps, indexed by $i, j \in I$, written briefly as $(\sigma_{ij})_{i,j\in I}$, forms *cocycles* of F *subordinate* to $(U_i)_{i\in I}$ if they fulfill the *cocycle conditions*:

(i) For all $i, j \in I$ and all $m \in U_i \cap U_j$, we have $\sigma_{ij}(m) = \sigma_{ji}(m)^{-1}$.
(ii) For all $i, j, k \in I$ and all $m \in U_i \cap U_j \cap U_k$, we have $\sigma_{ij}(m) \circ \sigma_{jk}(m) = \sigma_{ik}(m)$.

We use this to construct cocycles of fiber bundles, vector bundles and principal bundles.

REMARK 2.1.7. Let ξ be a fiber bundle, such that all of its fibers are biholomorphic to each other, and let F be a complex analytic manifold isomorphic to the fibers of ξ.

We choose an open covering $(U_i)_{i\in I}$ of M that trivializes ξ, i.e., for all $i \in I$, there exists $m_i \in U_i$ with a local trivialization $\Phi_{m_i} : p^{-1}(U_i) \to U_i \times \xi_{m_i}$. For all $i \in I$, we then compose Φ_{m_i} with a biholomorphism $U_i \times \xi_{m_i} \to U_i \times F$ to define $\Phi_i : p^{-1}(U_i) \to U_i \times F$, also called a local trivialization.

For all $i, j \in I$, we use the projection $\mathrm{pr}_2 : U_i \times F \to F$ to define:

$$\sigma_{ij} : U_i \cap U_j \to \mathrm{Aut}_{\mathrm{Hol}}(F), \quad m \mapsto [v \mapsto \mathrm{pr}_2 \circ \Phi_i \circ \Phi_j^{-1}(m, v)]. \tag{2.1.2}$$

It is easy to show that $(\sigma_{ij})_{i,j\in I}$ fulfills the cocycle conditions from Definition 2.1.6, and thus $(\sigma_{ij})_{i,j\in I}$ forms cocycles of F, which we call *cocycles* of ξ.

In the cases of vector bundles and principal-G-bundles, we want cocycles to preserve the respective structures on their fibers by mapping into groups isomorphic to subgroups of $\mathrm{Aut}_{\mathrm{Hol}}(F)$. This is possible due to *(ii)* of Definition 2.1.2 and *(ii)* of Definition 2.1.4:

(a) Let E be a vector bundle of rank r, we choose the fiber $F = \mathbb{C}^r$. Due to the linearity of the fibers of E, $\mathrm{Aut}_{\mathrm{Hol}}(\mathbb{C}^r)$ may be replaced with the subgroup $\mathbf{GL}(\mathbb{C}^r)$ of $\mathrm{Aut}_{\mathrm{Hol}}(\mathbb{C}^r)$.

    Through the canonical representation $\rho : \mathbf{GL}(r, \mathbb{C}) \to \mathbf{GL}(\mathbb{C}^r)$, we can view cocycles $(\sigma_{ij})_{i,j\in I}$ of E as maps into $\mathbf{GL}(r, \mathbb{C})$.
(b) Let ξ be a principal-G-bundle, we choose the fiber $F = G$. Due to the fibers of ξ being right-G-torsors, $\mathrm{Aut}_{\mathrm{Hol}}(G)$ may be replaced with the subgroup $\mathrm{Aut}_{G-\mathrm{equiv}}(G)$ of G-equivariant biholomorphisms from G to itself.

Since $\mathrm{Aut}_{G-\mathrm{equiv}}(G)$ is canonically isomorphic to G, we can view cocycles $(\sigma_{ij})_{i,j\in I}$ of ξ as maps into G.

The following lemma explains how cocycles characterize bundles up to isomorphism.

Lemma 2.1.8. *Let $(\sigma_{ij})_{i,j\in I}$ be cocycles of F.*

(i) The set of equivalence classes $\eta = \bigsqcup_{i\in I}(U_i \times F)/\sim$, with respect to the relation:

$$(m, v) \sim (u, w), \text{ if there exists } i, j \in I, \text{ such that } m = u \in U_i \cap U_j, v = \sigma_{ij}(w), \quad (2.1.3)$$

can be equipped with a unique complex analytic manifold structure, such that the natural map $q : \eta \to M$ is a fiber bundle.

(ii) The cocycles $(\sigma_{ij})_{i,j\in I}$ of F form cocycles of η.

(iii) Any fiber bundle $p : \xi \to M$, that admits the same cocycles $(\sigma_{ij})_{i,j\in I}$, is isomorphic to η as a fiber bundle.

These statements also hold when replacing fiber bundles with vector bundles or principal bundles, by setting $F = \mathbb{C}^r$ or $F = G$, and using (a) and (b) of Remark 2.1.7.

Proof. For *(i)* and *(ii)*, see the proof of [Ham17, Theorem 4.3.3]. This proof is written for smooth fiber bundles, but also works for holomorphic fiber bundles, vector bundles and principal bundles.

For *(iii)*, we consider local trivializations $\Phi_i : p^{-1}(U_i) \to U_i \times F$, $i \in I$, that induce $(\sigma_{ij})_{i,j\in I}$. For all $i \in I$, we define the fiber-preserving holomorphic map:

$$\varphi_i : p^{-1}(U_i) \to \eta, \quad v \mapsto [\Phi_i(v)]. \quad (2.1.4)$$

By construction of the equivalence relation $\sim$, the maps $(\varphi_i)_{i\in I}$ agree on their intersections, gluing to a morphism $\varphi : \xi \to \eta$ of fiber bundles.

Similarly, φ has an inverse morphism $\psi : \eta \to \xi$, induced and glued from the following fiber-preserving holomorphic maps, for $i \in I$:

$$\psi_i : U_i \times F \to \xi, \quad (m, v) \mapsto \Phi_i^{-1}(m, v). \quad (2.1.5)$$

Thus, as a fiber bundle, ξ is isomorphic to η.

The proof works analogously for vector bundles and principal bundles. □

This lemma helps us construct frame bundles of vector bundles, which are important examples of principal bundles.

Example 2.1.9. Let $p : E \to M$ be a vector bundle of rank r, with the local trivializations $\Phi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^r$, $i \in I$, inducing cocycles $(\sigma_{ij})_{i,j\in I}$ of E. We construct the *frame bundle* $\mathrm{Fr}(E)$ of E, which is a principal-$\mathbf{GL}(r, \mathbb{C})$-bundle with the underlying set:

$$\mathrm{Fr}(E) = \bigsqcup_{m\in M} \mathrm{Iso}_{\mathbb{C}}(\mathbb{C}^r, E_m), \quad (2.1.6)$$

and the natural map $s : \mathrm{Fr}(E) \to M$ mapping $f \in \mathrm{Iso}_{\mathbb{C}}(\mathbb{C}^r, E_m)$ to m.

Using Lemma 2.1.8, $(\sigma_{ij})_{i,j\in I}$ induces a principal-$\mathbf{GL}(r, \mathbb{C})$-bundle $q : \eta \to M$. Through the projection $\mathrm{pr}_2 : U_i \times \mathbb{C}^r \to \mathbb{C}^r$, $i \in I$, and the canonical representation $\rho : \mathbf{GL}(r, \mathbb{C}) \to \mathbf{GL}(\mathbb{C}^r)$, we define the fiber-preserving maps:

$$\varphi_i : s^{-1}(U_i) \to \eta, \quad f \mapsto [s(f), \rho^{-1}(\mathrm{pr}_2 \circ \Phi_i \circ f)]. \quad (2.1.7)$$

The maps φ_i, $i \in I$, glue together to a fiber-preserving bijection $\varphi : \mathrm{Fr}(E) \to \eta$, through which we can endow $\mathrm{Fr}(E)$ with a principal-$\mathbf{GL}(r, \mathbb{C})$-bundle structure, independent of the choice of cocycles $(\sigma_{ij})_{i,j\in I}$.

We also determine the induced right-$\mathbf{GL}(r, \mathbb{C})$-action on $\mathrm{Fr}(E)$. For all $m \in M$, the right-$\mathbf{GL}(r, \mathbb{C})$-action on η_m is given by:

$$\eta_m \times \mathbf{GL}(r, \mathbb{C}) \to \eta_m, \quad ([m, A], B) \mapsto [m, AB]. \quad (2.1.8)$$

Through the isomorphism $\varphi : \mathrm{Fr}(E) \to \eta$, this action induces the following right-$\mathbf{GL}(r, \mathbb{C})$-action on the fiber of $\mathrm{Fr}(E)$ at m:

$$\mathrm{Iso}_{\mathbb{C}}(\mathbb{C}^r, E_m) \times \mathbf{GL}(r, \mathbb{C}) \to \mathrm{Iso}_{\mathbb{C}}(\mathbb{C}^r, E_m), \quad (f, A) \mapsto [v \mapsto f(Av)]. \tag{2.1.9}$$

As seen in this construction, $\mathrm{Fr}(E)$ and E share the same cocycles when both are viewed as maps into $\mathbf{GL}(r, \mathbb{C})$, though the former is a principal-$\mathbf{GL}(r, \mathbb{C})$-bundle, whilst the latter is a vector bundle.

2.1.2. Subbundles and reductions

We now discuss subbundles of vector bundles and reductions of principal bundles, and wish to compare them using frame bundles. These constructions are necessary for defining slope-(semi)-stability and Ramanathan-(semi)-stability.

Definition 2.1.10. Let $p : E \to M$ be a vector bundle of rank r. A subset $F \subseteq E$ is a *subbundle* of E of rank s if:

(i) For all $m \in M$, $F_m = F \cap E_m$ is a complex subspace of E_m of dimension s.
(ii) There exists an open covering $(U_i)_{i \in I}$ of M, with local trivializations $\Phi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^r$, $i \in I$, of E, such that for all $i \in I$, we have:

$$p^{-1}(U_i \times \mathbb{C}^s) = \bigsqcup_{m \in U_i} F_m. \tag{2.1.10}$$

By construction, subbundles F of E of rank s form vector bundles of rank s.

Let ξ be a principal-G-bundle, and let H be a closed complex Lie subgroup of G.

Definition 2.1.11. Let F be a complex analytic manifold with a holomorphic left-H-action $\rho : H \to \mathrm{Aut}_{\mathrm{Hol}}(F)$. We define the left-$H$-action on $\xi \times F$:

$$H \times (\xi \times F) \to \xi \times F, \quad (h, (v, w)) \mapsto (vh^{-1}, \rho(h)(w)), \tag{2.1.11}$$

and denote the set of left-H-orbits of this action $H \backslash (\xi \times F)$ by $\xi(F)$, with the elements $[v, w]$. We also denote the set of right-H-orbits of the right-G-action on ξ by ξ/H, with the elements vH.

It can be shown that ξ/H is a fiber bundle, with fibers biholomorphic to G/H. Moreover, $\xi(F)$ is a fiber bundle on ξ/H, with fibers biholomorphic to F, called the *associated bundle* of ξ with respect to F.

Sometimes, we have $H = G$, and hence $\xi(F)$ is a fiber bundle on $\xi/H = M$. In this case, if $(\sigma_{ij})_{i,j \in I}$ are cocycles of ξ, then $(\rho \circ \sigma_{ij})_{i,j \in I}$ are cocycles of $\xi(F)$.

In special cases, the bundle $\xi(F)$ is a vector bundle or a principal bundle.

(a) If $F = V$ is a complex vector space of dimension r, and if $\rho : H \to \mathbf{GL}(V)$ is a representation, then $\xi(V)$ is a vector bundle on ξ/H of rank r.
(b) If $F = G'$ is a complex Lie group, and if $\rho : H \to \mathrm{Aut}_{G'-\mathrm{equiv}}(G')$, then $\xi(G')$ is a principal-G'-bundle on ξ/H.

We present some examples of associated bundles, including adjoint bundles, determinant bundles and induced vector bundles.

Example 2.1.12. (a) The adjoint representation $\mathrm{Ad} : G \to \mathbf{GL}(\mathfrak{g})$ induces the *adjoint bundle* $\mathrm{ad}(\xi) = \xi(\mathfrak{g})$, which is an associated bundle of ξ with respect to $\mathfrak{g}$. Since $\mathfrak{g}$ is a Lie algebra, $\mathrm{ad}(\xi)$ also forms a Lie algebra bundle.
(b) For a vector bundle E of rank r, the determinant $\det : \mathbf{GL}(r, \mathbb{C}) \to \mathbf{GL}(\mathbb{C})$ induces an associated bundle $\det(E) = (\mathrm{Fr}(E))(\mathbb{C})$ of $\mathrm{Fr}(E)$, called the *determinant bundle*, which is a vector bundle of rank 1, i.e., a line bundle.
(c) Let G be a closed complex Lie subgroup of $\mathbf{GL}(r, \mathbb{C})$, i.e., a matrix Lie group. The canonical representation $\rho : G \to \mathbf{GL}(\mathbb{C}^r)$ induces a vector bundle $E_\xi = \xi(\mathbb{C}^r)$ of rank r, which called the *induced* vector bundle of ξ.

(d) The complex analytic manifold G/H carries the canonical left-G-action, which induces an associated bundle $\xi(G/H)$ with respect to G/H.

It is easy to show that:

$$\varphi : \xi/H \to \xi(G/H), \quad vH \mapsto [v, eH], \tag{2.1.12}$$

defines an isomorphism of fiber bundles. Thus, ξ/H is isomorphic to an associated bundle of ξ with respect to G/H.

Using associated bundles, we define reductions and extensions of principal bundles.

Definition 2.1.13. (a) For a morphism $\varphi : G \to G'$ of complex Lie groups, the principal-G'-bundle $\xi(G')$ is the *extension* of ξ to G'.

(b) Let ξ_H be a principal-H-bundle, such that for the inclusion $H \hookrightarrow G$ inducing the extension $\xi_H(G)$, there exists an isomorphism $\varphi : \xi_H(G) \to \xi$. We call (ξ_H, φ) a *reduction pair* of ξ to H.

(c) Two reduction pairs (ξ_H, φ) and (ξ'_H, φ') are called *isomorphic*, if there exists an isomorphism $\xi_H \simeq \xi'_H$ of principal bundles, such that:

$$\xi_H \to \xi_H(G) \xrightarrow{\varphi} \xi \text{ is equal to } \xi_H \simeq \xi'_H \to \xi'_H(G) \xrightarrow{\varphi'} \xi. \tag{2.1.13}$$

The following lemma allows us to identify isomorphism classes of reduction pairs with sections.

Lemma 2.1.14. *The following are in correspondence:*

(i) Isomorphism classes of reduction pairs (ξ_H, φ) of ξ to H.
(ii) Sections $\sigma : M \to \xi/H$ of ξ/H.

Proof. For *(i)* to *(ii)*, an isomorphism class of reduction pairs (ξ_H, φ) induces a morphism $\xi_H \to \xi/H$. This morphism maps whole fibers of ξ_H to elements of ξ/H, hence $\xi_H \to \xi/H$ factorizes through a section $\sigma : M \to \xi/H$.

For *(ii)* to *(i)*, given $\sigma : M \to \xi/H$, consider the pullback bundle $\sigma^*\xi$, which is a principal-H-bundle where for all $m \in M$, the fiber is $(\sigma^*\xi)_m = \sigma(m)$.

We can construct an isomorphism $\varphi : (\sigma^*\xi)(G) \to \xi$, given on the fibers $m \in M$ by:

$$(\sigma^*\xi)(G)_m = H\backslash(\sigma(m) \times G) \to \xi_m, \quad [v, g] \mapsto vg. \tag{2.1.14}$$

such that $(\sigma^*\xi, \varphi)$ defines an isomorphism class.

As these two constructions are inverse to each other, the claim follows. ☐

In practice, we are only interested in isomorphism classes of reduction pairs of ξ to H, corresponding to sections $\sigma : M \to \xi/H$, which we call *reductions* $\sigma^*\xi$ of ξ to H.

Note that reductions induce the commutative diagram:

$$\begin{array}{ccc} \sigma^*\xi & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{\sigma} & \xi/H \end{array} . \tag{2.1.15}$$

The following lemma establishes a correspondence between certain reductions of principal-$\mathbf{GL}(r, \mathbb{C})$-bundles and nontrivial subbundles of their induced vector bundles.

Lemma 2.1.15. *Let ξ be a principal-$\mathbf{GL}(r, \mathbb{C})$-bundle, with the induced vector bundle E_ξ. In the notation of Subsection 1.2.1, let $s = 1, \ldots, r-1$, and let P_I be the maximal standard parabolic subgroup of $\mathbf{GL}(r, \mathbb{C})$, corresponding to $I = \{\alpha_{s,s+1}\} \subseteq \triangle$. The following are in correspondence:*

(i) Subbundles $F \neq 0, E_\xi$ of E_ξ of rank s.
(ii) Reductions $\sigma^*\xi$ of ξ to P_I.

PROOF. There exists a canonical isomorphism $\varphi : \xi \to \mathrm{Fr}(E_\xi)$ of principal bundles. Let $(e_1, \dots, e_r)$ be the canonical $\mathbb{C}$-basis of $\mathbb{C}^r$, then for all $m \in M$, φ is given by:

$$\xi_m \to \mathrm{Fr}(E_\xi)_m = \mathrm{Iso}_{\mathbb{C}}(\mathbb{C}^r, \mathbf{GL}(r,\mathbb{C})\backslash(\xi_m \times \mathbb{C}^r)), v \mapsto [e_i \mapsto [v, e_i]]. \tag{2.1.16}$$

We start from *(i)* to *(ii)*. From a subbundle $F \neq 0, E_\xi$ of E_ξ, we induce a section $\sigma' : M \to \mathrm{Fr}(E_\xi)/P_I$, $m \mapsto \sigma'(m) \in \mathrm{Iso}_{\mathbb{C}}(\mathbb{C}^r, (E_\xi)_m)/P_I$, where $\sigma'(m)$ represents the isomorphisms $\mathbb{C}^r \to (E_\xi)_m$ that restrict to $\mathbb{C}^s \to F_m$. Through $\varphi : \xi \to \mathrm{Fr}(E_\xi)$, σ' corresponds to a reduction $\sigma^*\xi$ of ξ to P_I.

For *(ii)* to *(i)*, σ induces a section $\sigma' : M \to \mathrm{Fr}(E_\xi)/P_I$ through φ. Then for all $m \in M$, $F_m = \sigma'(m)(\mathbb{C}^s)$ is a well-defined subspace of $(E_\xi)_m$. Since σ' is a section, the sets F_m, $m \in M$, glue to a subbundle $F \neq 0, E_\xi$ of E_ξ.

As these two constructions are inverse to each other, the claim follows. □

Analogous versions of Lemma 2.1.15 exist for different structure groups. We wish to present a special orthogonal version, for which we first need to construct special orthogonal vector bundles, which are orthogonal vector bundles that are special vector bundles.

For a line bundle $p : L \to M$ isomorphic to the product bundle $M \times \mathbb{C}$, global trivializations $\Phi : L \to M \times \mathbb{C}$ correspond to everywhere nonzero sections $\tau : M \to L$ as follows:

$$\{\text{Global trivializations of } L\} \leftrightarrow \{\text{Everywhere nonzero sections of } L\} \tag{2.1.17}$$

$$\Phi \mapsto \tau_\Phi = [m \mapsto \Phi^{-1}(m, 1)], \tag{2.1.18}$$

$$\Phi_\tau = [v \mapsto (p(v), \lambda), v = \lambda\tau(m)] \leftarrow\!\shortmid \tau. \tag{2.1.19}$$

Through this correspondence, we can define special vector bundles. Let E be a vector bundle of rank r.

DEFINITION 2.1.16. A *special vector bundle* (E, τ) consists of a vector bundle E, whose determinant bundle $\det(E)$ is isomorphic to the product bundle $M \times \mathbb{C}$, and an everywhere nonzero section $\tau : M \to \det(E)$.

Similarly to the frame bundles from Example 2.1.9, we can construct special frame bundles of special vector bundles.

REMARK 2.1.17. For a special vector bundle (E, τ), we can construct a *special frame bundle* $\mathrm{Fr}_{\mathbf{SL}}(E, \tau)$, which is a principal-$\mathbf{SL}(r,\mathbb{C})$-bundle, whose underlying set is:

$$\mathrm{Fr}_{\mathbf{SL}}(E, \tau) = \bigsqcup_{m \in M} \mathrm{SIso}^\tau_{\mathbb{C}}(\mathbb{C}^r, E_m), \tag{2.1.20}$$

where $\mathrm{SIso}^\tau_{\mathbb{C}}(\mathbb{C}^r, E_m) = \{f \in \mathrm{Iso}_{\mathbb{C}}(\mathbb{C}^r, E_m) | [f, 1] = \tau(m)\}$.

The right-$\mathbf{SL}(r,\mathbb{C})$-action on the fibers $m \in M$ is given by:

$$\mathrm{SIso}^\tau_{\mathbb{C}}(\mathbb{C}^r, E_m) \times \mathbf{SL}(r,\mathbb{C}) \to \mathrm{SIso}^\tau_{\mathbb{C}}(\mathbb{C}^r, E_m), \quad (f, A) \mapsto [v \mapsto f(Av)], \tag{2.1.21}$$

which is well-defined.

Before we construct orthogonal vector bundles, we remind ourselves of direct sums and dual bundles of vector bundles.

DEFINITION 2.1.18. Let E be a vector bundle of rank r, and let F be a vector bundle of rank s.

(a) The *direct sum* $E \oplus F$ is a vector bundle of rank $r + s$ with the underlying set $E \oplus F = \bigsqcup_{m \in M} E_m \oplus F_m$.

The local trivializations of $E \oplus F$ are direct sums of local trivializations of E and F.

(b) The *dual bundle* E^* of E is a vector bundle of rank r with the underlying set $E^* = \bigsqcup_{m\in M} E_m^\vee$.
The local trivializations of E^* are induced from local trivializations of E.

DEFINITION 2.1.19. (a) An *orthogonal vector bundle* (E, β) consists of a vector bundle E admitting a nondegenerate symmetric bilinear form $\beta : E \oplus E \to \mathbb{C}$, i.e., a holomorphic map such that for all $m \in M$, $\beta_m = \beta|_{E_m \oplus E_m}$ is a nondegenerate symmetric bilinear form.

(b) For an orthogonal vector bundle (E, β), a subbundle F of (E, β) is *(co)-isotropic*, if for all $m \in M$, F_m is a (co)-isotropic subspace of E_m, with respect to β_m.
A subbundle F of (E, β) is called *Lagrangian*, if F is isotropic and coisotropic.

REMARK 2.1.20. Let (E, β) be an orthogonal vector bundle.

(a) The vector bundle E is self-dual, as there exists an isomorphism of vector bundles:
$$E \to E^*, \quad v \mapsto \beta(\_, v). \tag{2.1.22}$$
(b) We can construct an *orthogonal frame bundle* $\mathrm{Fr}_{\mathbf{O}}(E, \beta)$, which is a principal-$\mathbf{O}(r, \mathbb{C})$-bundle, whose underlying set is:
$$\mathrm{Fr}_{\mathbf{O}}(E, \beta) = \bigsqcup_{m\in M} \mathrm{Orth}_{\mathbb{C}}^{\beta}(\mathbb{C}^r, E_m), \tag{2.1.23}$$
where $\mathrm{Orth}_{\mathbb{C}}^{\beta}(\mathbb{C}^r, E_m)$ consists of isomorphisms $f \in \mathrm{Iso}_{\mathbb{C}}(\mathbb{C}^r, E_m)$, such that for the standard nondegenerate symmetric bilinear form $\langle \_, \_ \rangle$ on $\mathbb{C}^r$, and for all $v, w \in \mathbb{C}^r$, we have:
$$\langle v, w \rangle = \beta_m(f(v), f(w)). \tag{2.1.24}$$

We can now combine orthogonal vector bundles and special vector bundles to construct special orthogonal vector bundles.

DEFINITION 2.1.21. A *special orthogonal vector bundle* (E, β, τ) consists of an orthogonal vector bundle (E, β) that is also a special vector bundle (E, τ).

By construction, special orthogonal vector bundles inherit all the properties of orthogonal vector bundles and special vector bundles.

REMARK 2.1.22. Let (E, β, τ) be a special orthogonal vector bundle. We can construct a *special orthogonal frame bundle* $\mathrm{Fr}_{\mathbf{SO}}(E, \beta, \tau)$, which is a principal-$\mathbf{SO}(r, \mathbb{C})$-bundle, whose underlying set is:
$$\mathrm{Fr}_{\mathbf{SO}}(E, \beta, \tau) = \bigsqcup_{m\in M} \mathrm{SOrth}_{\mathbb{C}}^{\beta,\tau}(\mathbb{C}^r, E_m), \tag{2.1.25}$$
where $\mathrm{SOrth}_{\mathbb{C}}^{\beta,\tau}(\mathbb{C}^r, E_m) = \mathrm{Orth}_{\mathbb{C}}^{\beta}(\mathbb{C}^r, E_m) \cap \mathrm{SIso}_{\mathbb{C}}^{\tau}(\mathbb{C}^r, E_m)$.

We now present the special orthogonal version of Lemma 2.1.15.

LEMMA 2.1.23. *Let $r \geq 3$, and let ξ be a principal-$\mathbf{SO}(r, \mathbb{C})$-bundle. The induced vector bundle E_ξ admits the structure of a special orthogonal vector bundle (E_ξ, β, τ). We write $r = 2n + 1$ if r is odd, and $r = 2n$ if r is even. In the notation of Subsection 1.2.2, let $s = 1, \ldots, n$, and let P_I be the standard parabolic subgroup of $\mathbf{SO}(r, \mathbb{C})$, corresponding to:*

$$I = \{e_s - e_{s+1}\} \subseteq \triangle, \qquad s \neq n-1, n, \tag{2.1.26}$$
$$I = \{e_{n-1} - e_n\} \subseteq \triangle, \qquad s = n-1, \qquad r \textit{ is odd}, \tag{2.1.27}$$
$$I = \{e_n\} \subseteq \triangle, \qquad s = n, \qquad r \textit{ is odd}, \tag{2.1.28}$$
$$I = \{e_n - e_{n+1}, e_n + e_{n+1}\} \subseteq \triangle, \qquad s = n-1, \qquad r \textit{ is even}, \tag{2.1.29}$$

$$I = \{e_{n-1} + e_n\} \subseteq \triangle, \qquad s = n, \qquad r \text{ is even.} \tag{2.1.30}$$

The following are in correspondence:

(i) *Isotropic subbundles $F \neq 0$ of (E_ξ, β) of rank s.*
(ii) *Reductions $\sigma^*\xi$ of ξ to P_I.*

PROOF. Similarly to the proof of Lemma 2.1.15, there exists a canonical isomorphism $\varphi : \xi \to \mathrm{Fr}_{\mathbf{SO}}(E_\xi, \beta, \tau)$ of principal bundles.

Using Remark 1.2.6, P_I is the stabilizer of a flag:

$$\mathcal{F} : 0 = V_0 \subsetneq V_1 \subseteq V_1^\perp \subsetneq V_0^\perp = \mathbb{C}^r, \tag{2.1.31}$$

such that $\dim_{\mathbb{C}}(V_1) = s$ and $\dim_{\mathbb{C}}(V_1^\perp) = r - s$.

We start from *(i)* to *(ii)*. From an isotropic subbundle $F \neq 0$ of (E_ξ, β), we induce a section $\sigma' : M \to \mathrm{Fr}_{\mathbf{SO}}(E_\xi, \beta, \tau)/P_I$, $m \mapsto \sigma'(m) \in \mathrm{SOrth}_{\mathbb{C}}^{\beta,\tau}(\mathbb{C}^r, (E_\xi)_m)/P_I$, where $\sigma'(m)$ represents the special orthogonal maps $\mathbb{C}^r \to (E_\xi)_m$ that restrict to $V_1^\perp \to F_m^\perp$ and further to $V_1 \to F_m$. Through $\varphi : \xi \to \mathrm{Fr}_{\mathbf{SO}}(E_\xi, \beta, \tau)$, σ' corresponds to a reduction $\sigma^*\xi$ of ξ to P_I.

For *(ii)* to *(i)*, σ induces a section $\sigma' : M \to \mathrm{Fr}_{\mathbf{SO}}(E_\xi, \beta, \tau)/P_I$ through φ. Then for all $m \in M$, $F_m = \sigma'(m)(V_1)$ is a well-defined isotropic subspace of $(E_\xi)_m$, since:

$$F_m = \sigma'(m)(V_1) \subseteq \sigma'(m)(V_1^\perp) = \sigma'(m)(V_1)^\perp = F_m^\perp. \tag{2.1.32}$$

Since σ' is a section, the sets F_m, $m \in M$, glue to an isotropic subbundle $F \neq 0$ of (E_ξ, β).

As these two constructions are inverse to each other, the claim follows. □

In the case where ξ is a principal-$\mathbf{SO}(2, \mathbb{C})$-bundle, this correspondence cannot apply, as there exists no maximal parabolic subgroups of $\mathbf{SO}(2, \mathbb{C})$. Instead, nontrivial isotropic subbundles of (E_ξ, β) directly correspond to sections σ of ξ.

Finally, we construct a symplectic analog of Lemma 2.1.15 and Lemma 2.1.23.

DEFINITION 2.1.24. Let E be a vector bundle of rank $2n$.

(a) A *symplectic vector bundle* (E, β) consists of a vector bundle E admitting a symplectic form $\beta : E \oplus E \to \mathbb{C}$, i.e., a holomorphic map such that for all $m \in M$, $\beta_m = \beta|_{E_m \oplus E_m}$ is a symplectic form.
(b) For a symplectic vector bundle (E, β), a subbundle F of (E, β) is *(co)-isotropic*, if for all $m \in M$, F_m is a (co)-isotropic subspace of E_m, with respect to β_m.
A subbundle F of (E, β) is called *Lagrangian*, if F is isotropic and coisotropic.

REMARK 2.1.25. Let (E, β) be a symplectic vector bundle of rank $2n$.

(a) The vector bundle E is self-dual, as there exists an isomorphism of vector bundles:

$$E \to E^*, \quad v \mapsto \beta(\_, v). \tag{2.1.33}$$

(b) We can construct a *symplectic frame bundle* $\mathrm{Fr}_{\mathbf{Sp}}(E, \beta)$, which is a principal-$\mathbf{Sp}(2n, \mathbb{C})$-bundle, whose underlying set is:

$$\mathrm{Fr}_{\mathbf{Sp}}(E, \beta) = \bigsqcup_{m \in M} \mathrm{Sp}_{\mathbb{C}}^{\beta}(\mathbb{C}^{2n}, E_m), \tag{2.1.34}$$

whereby $\mathrm{Sp}_{\mathbb{C}}^{\beta}(\mathbb{C}^{2n}, E_m)$ consists of isomorphisms f, such that for the standard symplectic form $\langle \_, \_ \rangle$ on $\mathbb{C}^{2n}$, and for all $v, w \in \mathbb{C}^{2n}$, we have:

$$\langle v, w \rangle = \beta_m(f(v), f(w)). \tag{2.1.35}$$

LEMMA 2.1.26. *Let ξ be a principal-$\mathbf{Sp}(2n, \mathbb{C})$-bundle. The induced vector bundle E_ξ admits the structure of a symplectic vector bundle (E_ξ, β). In the notation of Subsection*

1.2.3, let $s = 1, \dots, n$, and let P_I be the maximal standard parabolic subgroup of $\mathbf{Sp}(2n, \mathbb{C})$, corresponding to:

$$I = \{e_s - e_{s+1}\} \subseteq \triangle, \qquad s \neq n, \tag{2.1.36}$$

$$I = \{2e_n\} \subseteq \triangle, \qquad s = n. \tag{2.1.37}$$

The following are in correspondence:

- *(i) Isotropic subbundles $F \neq 0$ of (E_ξ, β) of rank s.*
- *(ii) Reductions $\sigma^*\xi$ of ξ to P_I.*

Proof. Analogous to that of Lemma 2.1.23, where we replace special orthogonal frame bundles with symplectic frame bundles. □

2.2. Stability in the sense of Ramanathan

We want to define notions of slope-(semi)-stability of vector bundles, and of Ramanathan-(semi)-stability of principal bundles. Using the correspondences we found between subbundles of vector bundles and reductions of frame bundles, we can compare these two notions of stability.

2.2.1. Stability in the sense of Ramanathan

For these definitions of stability, we must first define degrees and slopes of vector bundles, for which our setting needs an extra assumption on the base space. Unless otherwise stated, bundles are now constructed on a compact connected Riemann surface X.

Definition 2.2.1. (a) Let $\sigma : X \to L$ a nontrivial meromorphic section of a line bundle. We define the *zeros* of σ as $\mathbf{Z}(\sigma) = \{x \in X | \sigma(x) = 0\}$, and the *poles* of σ as $\mathbf{P}(\sigma) = \{x \in X | (1/\sigma)(x) = 0\}$.

The *order* of σ at $x \in \mathbf{Z}(\sigma)$ is the multiplicity of the zero of σ at x, and at $x \in \mathbf{P}(\sigma)$, it is the multiplicity of the zero of $(1/\sigma)$ at x.

The *degree* of σ is defined as:

$$\deg(\sigma) = \sum_{x \in \mathbf{Z}(\sigma)} \mathrm{ord}_x(\sigma) - \sum_{x \in \mathbf{P}(\sigma)} \mathrm{ord}_x(\sigma). \tag{2.2.1}$$

(b) Let $E \neq 0$ be a vector bundle of rank r. The *degree* $\deg(E)$ of E is the degree $\deg(\sigma)$ of a nontrivial meromorphic section σ of the determinant bundle $\det(E)$ of E.

These definitions are well-defined, since it can be shown that:

- (a) The degree in (2.2.1) is a well-defined integer, since X is compact.
- (b) For a line bundle L on a compact connected Riemann surface, nontrivial meromorphic sections are guaranteed to exist.
- (c) For two nontrivial meromorphic sections σ and σ' on the same line bundle L, we have $\deg(\sigma) = \deg(\sigma')$.

Degrees store useful topological properties of vector bundles, intuitively speaking, how they globally "twist". We now state the additivity of degrees of vector bundles, using tensor bundles.

Definition 2.2.2. Let E be a vector bundle of rank r, and let F be a vector bundle of rank s. The *tensor bundle* $E \otimes F$ of E is a vector bundle of rank rs with the underlying set $E \otimes F = \bigsqcup_{m \in M} E_m \otimes F_m$.

The local trivializations of $E \otimes F$ are induced from local trivializations of E and F.

REMARK 2.2.3. For a short exact sequence of nontrivial vector bundles:

$$0 \to E' \to E \to E'' \to 0, \tag{2.2.2}$$

we have $\deg(E) = \deg(E') + \deg(E'')$, which is called the *additivity* of degrees of vector bundles. This is proven by showing that there exists an isomorphism:

$$\det(E) \simeq \det(E') \otimes \det(E''), \tag{2.2.3}$$

of determinant bundles, and then comparing meromorphic sections between $\det(E)$ and $\det(E') \otimes \det(E'')$.

Using degrees of vector bundles, we can define slopes, leading to the notion of slope-(semi)-stability, from [Mum62].

DEFINITION 2.2.4. Let $E \neq 0$ be a vector bundle of rank r.

(a) The *slope* of E is $\mu(E) = \deg(E)/r$.
(b) The bundle E is *slope-stable* if all subbundles $F \neq 0, E$ of E fulfill $\mu(F) < \mu(E)$.
(c) The bundle E is *slope-semistable* if all subbundles $F \neq 0, E$ of E fulfill $\mu(F) \leq \mu(E)$.

It is clear that slope-stability implies slope-semistability.

As an example, line bundles are slope-stable since they have no nontrivial subbundles.

We now define Ramanathan-(semi)-stability for principal bundles, as in [Ram75]. Let G be a connected complex reductive group and let ξ be a principal-G-bundle. We investigate the adjoint representation $\mathrm{Ad} : G \to \mathbf{GL}(\mathfrak{g})$.

REMARK 2.2.5. For a parabolic subgroup P of G, let $\sigma^*\xi$ be a reduction of ξ to P. Since P is a complex algebraic subgroup of G, the adjoint representation $\mathrm{Ad} : P \to \mathbf{GL}(\mathfrak{g})$ induces representations $\mathrm{Ad} : P \to \mathbf{GL}(\mathfrak{p})$ and $\mathrm{Ad} : P \to \mathbf{GL}(\mathfrak{g}/\mathfrak{p})$, inducing the associated vector bundles $(\sigma^*\xi)(\mathfrak{g})$, $(\sigma^*\xi)(\mathfrak{p})$ and $(\sigma^*\xi)(\mathfrak{g}/\mathfrak{p})$. For these vector bundles, we have that:

(a) There exists an isomorphism $\varphi : (\sigma^*\xi)(\mathfrak{g}) \to \mathrm{ad}(\xi)$, given on the fibers $x \in X$ by:

$$(\sigma^*\xi)(\mathfrak{g})_x = P\backslash(\sigma(x) \times \mathfrak{g}) \to \mathrm{ad}(\xi)_x = G\backslash(\xi_x \times \mathfrak{g}), \quad [v, Y] \mapsto [v, Y]. \tag{2.2.4}$$

(b) By definition, $(\sigma^*\xi)(\mathfrak{p})$ is the adjoint bundle $\mathrm{ad}(\sigma^*\xi)$.
(c) The derivative of the fiber bundle $p : \xi/P \to X$ is a morphism $Dp : T(\xi/P) \to TX$ of vector bundles of full rank, with a change of base space:

$$\begin{array}{ccc} T(\xi/P) & \xrightarrow{Dp} & TX \\ \downarrow & & \downarrow \\ \xi/P & \xrightarrow{p} & X \end{array} . \tag{2.2.5}$$

The kernel $V_{\xi/P} = \ker(Dp)$ of Dp is a well-defined subbundle of $T(\xi/P)$ on the base space ξ/P, called the *vertical tangent* subbundle of ξ/P.

In the proof of [Ram75, Lemma 2.1], it is stated that $V_{\xi/P}$ is isomorphic to the associated bundle $\xi(\mathfrak{g}/\mathfrak{p})$ on ξ/P, induced by $\mathrm{Ad} : P \to \mathbf{GL}(\mathfrak{g}/\mathfrak{p})$. Thus, through the reduction $\sigma : X \to \xi/P$, we obtain a vector bundle $\sigma^*V_{\xi/P}$ isomorphic to $\sigma^*(\xi(\mathfrak{g}/\mathfrak{p})) \simeq (\sigma^*\xi)(\mathfrak{g}/\mathfrak{p})$.

We can now define Ramanathan-(semi)-stability, as seen in [Ram75, Definition 1.1].

DEFINITION 2.2.6. The bundle ξ is *Ramanathan-(semi)-stable*, if for all maximal parabolic subgroups P of G, and for all reductions $\sigma^*\xi$ of ξ to P, we have $\deg(\sigma^*V_{\xi/P})(\geq)0$.

In order to verify Ramanathan-(semi)-stability, it suffices to only verify the inequality for reductions to maximal standard parabolic subgroups of G, with respect to a fixed Borel subgroup B of G. This is due to the conjugacy of parabolic subgroups from Remark 1.1.26.

We aim to prove that Ramanathan-(semi)-stability can also be equivalently verified through the inequality $\deg(\mathrm{ad}(\sigma^*\xi))(\leq)0$, for which we need the following remark.

Remark 2.2.7. Let $\sigma^*\xi$ be a reduction of ξ to a parabolic subgroup P of G.

(a) We observe the short exact sequence of complex vector spaces:

$$0 \to \mathfrak{p} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{p} \to 0. \tag{2.2.6}$$

Since $\mathrm{ad}(\sigma^*\xi)$, $\mathrm{ad}(\xi)$ and $\sigma^*V_{\xi/P}$ are isomorphic to associated bundles of $\sigma^*\xi$, with respect to the adjoint representation of P acting on (2.2.6), we obtain the short exact sequence of vector bundles:

$$0 \to \mathrm{ad}(\sigma^*\xi) \to \mathrm{ad}(\xi) \to \sigma^*V_{\xi/P} \to 0. \tag{2.2.7}$$

(b) Since $\mathfrak{g}$ is reductive, Theorem 1.1.5 implies that there exists a nondegenerate symmetric ad-invariant bilinear form on $\mathfrak{g}$, which is thus also Ad-invariant.

Using that $\mathrm{ad}(\xi)$ is induced by the adjoint representation $\mathrm{Ad} : G \to \mathbf{GL}(\mathfrak{g})$, this bilinear form induces a nondegenerate symmetric bilinear form β on $\mathrm{ad}(\xi)$, such that $(\mathrm{ad}(\xi), \beta)$ is an orthogonal vector bundle.

Remark 2.2.8. Let $E \neq 0$ be a vector bundle of rank r. By verifying that $\det(E) \otimes \det(E^*)$ is isomorphic to the product bundle $X \times \mathbb{C}$, we have that $\deg(E) = -\deg(E^*)$.

Lemma 2.2.9. *The following are equivalent:*

(i) The bundle ξ is Ramanathan-(semi)-stable.

(ii) For all maximal parabolic subgroups P of G, and for all reductions $\sigma^*\xi$ of ξ to P, we have $\deg(\mathrm{ad}(\sigma^*\xi))(\leq)0$.

Proof. Due to the additivity of degrees from Remark 2.2.3, we have:

$$\deg(\mathrm{ad}(\xi)) = \deg(\mathrm{ad}(\sigma^*\xi)) + \deg(\sigma^*V_{\xi/P}), \tag{2.2.8}$$

from the short exact sequence in (2.2.7).

Since $\mathrm{ad}(\xi)$ is self-dual due to (a) of Remark 2.1.20, we use that degrees are invariant under isomorphisms to imply $\deg(\mathrm{ad}(\xi)) = -\deg(\mathrm{ad}(\xi))$, using Remark 2.2.8. Thus, $\deg(\mathrm{ad}(\xi)) = 0$, and $\deg(\mathrm{ad}(\sigma^*\xi)) = -\deg(\sigma^*V_{\xi/P})$, from which the equivalence is clear. □

We now want to compare Ramanathan-(semi)-stability for the structure groups $\mathbf{GL}(r,\mathbb{C})$, $\mathbf{SO}(r,\mathbb{C})$ and $\mathbf{Sp}(2n,\mathbb{C})$, to slope-(semi)-stability.

2.2.2. The case of $\mathbf{GL}(r,\mathbb{C})$

For a principal-$\mathbf{GL}(r,\mathbb{C})$-bundle ξ, we aim to prove that the correspondence in Lemma 2.1.15 preserves the notions of slope-(semi)-stability and Ramanathan-(semi)-stability, following [HM04, Proposition 1] and [HM04, Corollary 1].

Before stating this result, we need to find an appropriate isomorphism on vertical tangent bundles. For this, we construct quotient bundles and their cocycles.

Let $E \neq 0$ be a vector bundle of rank r.

Definition 2.2.10. For a subbundle F of $p : E \to X$ of rank s, the *quotient bundle* E/F is a vector bundle of rank $r - s$ with the underlying set $E/F = \bigsqcup_{m\in M} E_m/F_m$.

For the local trivializations Φ_i, $i \in I$, of E, from (a) of Definition 2.1.10, we define local trivializations Ψ_i, $i \in I$, of E/F, using the projection $\mathrm{pr}_2 : U_i \times \mathbb{C}^r \to \mathbb{C}^r$:

$$\Psi_i : \bigsqcup_{m\in U_i} E_m/F_m \to U_i \times (\mathbb{C}^r/\mathbb{C}^s), \quad [v] \mapsto (p(v), [\mathrm{pr}_2 \circ \Phi_i(v)]). \tag{2.2.9}$$

From the proof of Lemma 2.1.15, we can relate cocycles of vector bundles with cocycles of their subbundles and quotients.

REMARK 2.2.11. Let F be a subbundle of E of rank s. There exists cocycles $(\sigma_{ij})_{i,j\in I}$ of E mapping into P_I, where $I = \{\alpha_{s,s+1}\} \subset \triangle$:

$$\sigma_{ij} : U_i \cap U_j \to P_I, m \mapsto \begin{pmatrix} \alpha_{ij}(m) & \beta_{ij}(m) \\ 0 & \delta_{ij}(m) \end{pmatrix}, \quad i,j \in I, \tag{2.2.10}$$

and $(\alpha_{ij})_{i,j\in I}$ forms cocycles of F, and $(\delta_{ij})_{i,j\in I}$ forms cocycles of E/F.

Note that if we are first given the existence of cocycles $(\sigma_{ij})_{i,j\in I}$ of E in the form of (2.2.10), we can also induce a subbundle F of E of rank s, admitting the cocycles $(\alpha_{ij})_{i,j\in I}$.

LEMMA 2.2.12. *Let ξ be a principal-$\mathbf{GL}(r,\mathbb{C})$-bundle. In the notation of Subsection 1.2.1, let $s = 1,\dots,r-1$, and let P_I be the maximal standard parabolic subgroup of $\mathbf{GL}(r,\mathbb{C})$, corresponding to $I = \{\alpha_{s,s+1}\} \subseteq \triangle$.*

Let $\sigma^*\xi$ be any reduction of ξ to P_I, and let $F \neq 0, E_\xi$ be the corresponding subbundle of E_ξ of rank s, through Lemma 2.1.15. We claim that:

$$\sigma^* V_{\xi/P_I} \simeq F^* \otimes (E_\xi/F), \tag{2.2.11}$$

as vector bundles.

PROOF. Let $(\sigma_{ij})_{i,j\in I}$ be cocycles of $\sigma^*\xi$ and E_ξ, which map into P_I, in the form of (2.2.10), such that $(\alpha_{ij})_{i,j\in I}$ forms cocycles of F and $(\delta_{ij})_{i,j\in I}$ forms cocycles of E_ξ/F. To find cocycles $(\tau_{ij})_{i,j\in I}$ of $\sigma^* V_{\xi/P_I}$, we compose $(\sigma_{ij})_{i,j\in I}$ with $\mathrm{Ad} : P_I \to \mathbf{GL}(\mathfrak{gl}(r,\mathbb{C})/\mathfrak{p}_I)$. Firstly, we calculate:

$$\mathrm{Ad}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} * & * \\ C & * \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} * & * \\ C & * \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1}, \tag{2.2.12}$$

$$= \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} * & * \\ C & * \end{pmatrix}\begin{pmatrix} a^{-1} & * \\ 0 & d^{-1} \end{pmatrix}, \tag{2.2.13}$$

$$= \begin{pmatrix} * & * \\ dCa^{-1} & * \end{pmatrix}. \tag{2.2.14}$$

Hence, the cocycles $(\tau_{ij})_{i,j\in I}$ are of the form $x \mapsto [C \mapsto \delta_{ij}(x) C \alpha_{ij}(x)^{-1}]$, otherwise written $x \mapsto (\alpha_{ij}(x)^{-1})^T \otimes \delta_{ij}(x)$, where $\otimes$ denotes the Kronecker-product of matrices. Following [GH94, 0.5], these are cocycles of $F^* \otimes (E_\xi/F)$, so the claim of the lemma follows. ☐

With this isomorphism, it becomes easy to translate between Ramanathan-(semi)-stability and slope-(semi)-stability.

We now use the following results on degrees of vector bundles. Let $E \neq 0$ be a vector bundle of rank r.

REMARK 2.2.13. (a) Using the additivity of degrees, if F is a subbundle of E, then $\deg(E/F) = \deg(E) - \deg(F)$.
(b) For a vector bundle $F \neq 0$ of rank s, we have $\deg(E \otimes F) = \deg(E)s + \deg(F)r$.

THEOREM 2.2.14. *Let ξ be a principal-$\mathbf{GL}(r,\mathbb{C})$-bundle. The following are equivalent:*
(i) The principal bundle ξ is Ramanathan-(semi)-stable.
(ii) The induced vector bundle E_ξ is slope-(semi)-stable.

PROOF. Any subbundle $F \neq 0, E_\xi$ of E_ξ of rank $s = 1,\dots,r-1$, corresponds to a reduction $\sigma^*\xi$ of ξ, through Lemma 2.1.15. Due to Lemma 2.2.12, we have:

$$\deg(\sigma^* V_{\xi/P_I}) = \deg(F^* \otimes (E_\xi/F)), \tag{2.2.15}$$

using properties of degrees from Remark 2.2.13, we have:

$$= -\deg(F)(r-s) + \deg(E_\xi/F)s, \tag{2.2.16}$$
$$= -\deg(F)(r-s) + \deg(E_\xi)s - \deg(F)s, \tag{2.2.17}$$
$$= -\deg(F)r + \deg(E_\xi)s. \tag{2.2.18}$$

Thus, $\deg(\sigma^* V_{\xi/P_I})(\geq)0$ is equivalent to $\mu(F)(\leq)\mu(E_\xi)$, so the theorem follows. □

2.2.3. The cases of $\mathbf{SO}(r,\mathbb{C})$ and $\mathbf{Sp}(2n,\mathbb{C})$

We also want special orthogonal and symplectic versions of Theorem 2.2.14, though proving this is more involved, since isomorphisms analogous to (2.2.11) of Lemma 2.2.12 are harder to describe. We start with the special orthogonal case, and state some results about (co)-isotropic subbundles of special orthogonal vector bundles.

For this subsection, let $r \geq 2$, and write $r = 2n+1$ if r is odd, and $r = 2n$ if r is even.

Lemma 2.2.15. *Let (E,β,τ) be a special orthogonal vector bundle of rank r. For all isotropic subbundles F of (E,β) of rank $s = 1,\dots,n$, we denote:*

$$F^\perp = \bigsqcup_{x\in X} F_x^\perp. \tag{2.2.19}$$

(i) The set $F^\perp$ is a coisotropic subbundle of (E,β) of rank $r-s$.
(ii) The bilinear form β and the section τ restricts to β_F and τ_F, such that $(F^\perp/F,\beta_F,\tau_F)$ is a special orthogonal vector bundle of rank $r-2s$.
Furthermore, F is isomorphic to the dual bundle $(E/F^\perp)^*$.
(iii) If $F \neq 0$, we have:

$$\mathit{deg}(F) = -\mathit{deg}(E/F^\perp) = -\mathit{deg}(E/F) = \mathit{deg}(F^\perp). \tag{2.2.20}$$

Proof. For *(i)* and *(ii)*, the claim is trivial for $F=0$ or $r=2$, so we assume $F\neq 0$ and $r\geq 3$.

Due to Lemma 2.1.23, F corresponds to a parabolic reduction $\sigma^*\mathrm{Fr}_{\mathbf{SO}}(E,\beta,\tau)$ of $\mathrm{Fr}_{\mathbf{SO}}(E,\beta,\tau)$, which induces $F^\perp$ as coisotropic subbundle of E, proving *(i)*.

We now prove *(ii)*. The parabolic reduction $\sigma^*\mathrm{Fr}_{\mathbf{SO}}(E,\beta,\tau)$ induces cocycles $(\sigma_{ij})_{i,j\in I}$ of E mapping into P_I, and through the isomorphism $P_I \simeq \overline{P}_I$, we also induce cocycles $(\overline{\sigma}_{ij})_{i,j\in I}$ of E mapping into $\overline{P}_I$.

The group $\overline{P}_I$ consists of matrices of the form:

$$M_{a,b,c,d} = \begin{pmatrix} a & b & c \\ 0 & d & -K_{r-2s}(d^{-1})^T(b^T)(a^{-1})^T K_s \\ 0 & 0 & K_s(a^{-1})^T K_s \end{pmatrix}, \tag{2.2.21}$$

with $a\in \mathbf{GL}(s,\mathbb{C})$, $b\in \mathrm{Mat}(s\times(r-2s),\mathbb{C})$, $c\in\mathrm{Mat}(s\times s,\mathbb{C})$ and $d\in\overline{\mathbf{SO}}(r-2s,\mathbb{C})$, such that $K_s(a^{-1})c + c^T K_s(a^{-1})^T K_s + K_s(a^{-1})bK_{r-2s}b^T(a^{-1})^T K_s = 0$. Due to this, $(\overline{\sigma}_{ij})_{i,j\in I}$ is of the form:

$$\overline{\sigma}_{ij} : U_i\cap U_j \to \overline{P}_I, m\mapsto \begin{pmatrix} \alpha_{ij}(m) & \beta_{ij}(m) & \dots \\ 0 & \delta_{ij}(m) & \dots \\ 0 & 0 & K_s(\alpha_{ij}(m)^{-1})^T K_s \end{pmatrix}, \tag{2.2.22}$$

where $(\alpha_{ij})_{i,j\in I}$ form cocycles of F, $(\delta_{ij})_{i,j\in I}$ form cocycles of $F^\perp/F$, $K_s(\alpha_{ij}(m)^{-1})^T K_s$ form cocycles of $E/F^\perp$, and cocycles of $F^\perp$ are of the form (2.2.10) from Remark 2.2.11.

Through these cocycles, we find canonical isomorphisms $F\simeq (E/F^\perp)^*$ and $\det(E)\simeq \det(F^\perp/F)$, inducing β_F and τ_F, such that $(F^\perp/F,\beta_F,\tau_F)$ is a special orthogonal vector bundle of rank $r-2s$.

We now prove *(iii)*. Since $F^\perp/F$ is self-dual, the additivity of degrees of vector bundles implies that $\deg(F) = \deg(F^\perp) + \deg(F^\perp/F) = \deg(F^\perp)$. Since E is also self-dual, the equality:

$$\deg(E) = \deg(F) + \deg(F^\perp/F) + \deg(E/F^\perp), \tag{2.2.23}$$

implies $\deg(F) = -\deg(E/F^\perp)$. Furthermore, we have:

$$\deg(E/F) = \deg(E/F^\perp) + \deg(F^\perp/F) = \deg(E/F^\perp). \tag{2.2.24}$$

□

We can now present a special orthogonal version of Lemma 2.2.12.

LEMMA 2.2.16. *Let $r \geq 3$, and let ξ be a principal-$\mathbf{SO}(r,\mathbb{C})$-bundle. The induced vector bundle E_ξ admits the structure of a special orthogonal vector bundle (E_ξ, β, τ). We write $r = 2n+1$ if r is odd, and $r = 2n$ if r is even.*

Let $F \neq 0$ be an isotropic subbundle of (E_ξ, β) of rank $s = 1, \dots, n$. Through Lemma 2.1.23, F induces a reduction of ξ to a standard parabolic subgroup P_I of $\mathbf{SO}(r,\mathbb{C})$, in the notation of Subsection 1.2.2, corresponding to $I \subseteq \triangle$. We claim that:

$$\begin{aligned} &det(\sigma^* V_{\xi/P_I}) \simeq det(F^* \otimes (F^\perp/F)) \otimes (det(F^*)^{\otimes(s-1)}), && F \neq F^\perp, && (2.2.25)\\ &det(\sigma^* V_{\xi/P_I}) \simeq det(F^*)^{\otimes(s-1)}, && F = F^\perp, && (2.2.26)\end{aligned}$$

where $(F^*)^{\otimes(s-1)}$ denotes the vector bundle F^* tensored with itself $s-1$-times.

PROOF. Since $\mathbf{SO}(r,\mathbb{C}) \simeq \overline{\mathbf{SO}}(r,\mathbb{C})$, we induce a principal-$\overline{\mathbf{SO}}(r,\mathbb{C})$-bundle $\overline{\xi} = \xi(\overline{\mathbf{SO}}(r,\mathbb{C}))$ as an extension of ξ. The reduction $\sigma^*\xi$ of ξ induces a reduction $\overline{\sigma}^*\overline{\xi}$ of $\overline{\xi}$ to $\overline{P}_I$. By composing cocycles $(\overline{\sigma}_{ij})_{i,j\in I}$ of $\overline{\sigma}^*\overline{\xi}$ with the adjoint representation $\mathrm{Ad} : \overline{P}_I \to \mathbf{GL}(\overline{\mathfrak{so}}(r,\mathbb{C})/\overline{\mathfrak{p}}_I)$, we obtain cocycles $(\overline{\tau}_{ij})_{i,j\in I}$ of $\overline{\sigma}^* V_{\overline{\xi}/\overline{P}_I}$, which is isomorphic to $\sigma^* V_{\xi/P_I}$ as a vector bundle.

In order to determine $(\det\circ\overline{\tau}_{ij})_{i,j\in I}$, we want to calculate $\mathrm{Ad} : \overline{P}_I \to \mathbf{GL}(\overline{\mathfrak{so}}(r,\mathbb{C})/\overline{\mathfrak{p}}_I)$. We first handle the case $F \neq F^\perp$. The complex vector space $\overline{\mathfrak{so}}(r,\mathbb{C})/\overline{\mathfrak{p}}_I$ consists of classes of matrices:

$$N_{D,G} = \begin{pmatrix} * & * & * \\ D & * & * \\ G & -K_s D^T K_{r-2s} & * \end{pmatrix}, \tag{2.2.27}$$

with $D \in \mathrm{Mat}((r-2s)\times s, \mathbb{C})$ and $G \in \overline{\mathfrak{so}}(s,\mathbb{C})$. Thus, we have $\overline{\mathfrak{so}}(r,\mathbb{C})/\overline{\mathfrak{p}}_I \simeq \mathrm{Mat}((r-2s)\times s,\mathbb{C}) \times \overline{\mathfrak{so}}(s,\mathbb{C})$ as complex vector spaces. We can then calculate $\mathrm{Ad} : \overline{P}_I \to \mathbf{GL}(\overline{\mathfrak{so}}(r,\mathbb{C})/\overline{\mathfrak{p}}_I)$, using the description of matrices $M_{a,b,c,d}$ in $\overline{P}_I$ from (2.2.21):

$$\mathrm{Ad}(M_{a,b,c,d})(N_{D,G}) \tag{2.2.28}$$

$$= \begin{pmatrix} \cdots & \cdots & \cdots \\ 0 & d & -K_{r-2s}(d^{-1})^T(b^T)(a^{-1})^T K_s \\ 0 & 0 & K_s(a^{-1})^T K_s \end{pmatrix} \begin{pmatrix} * & * & * \\ D & * & * \\ G & \cdots & * \end{pmatrix} \begin{pmatrix} a^{-1} & \cdots & \cdots \\ 0 & \cdots & \cdots \\ 0 & 0 & \cdots \end{pmatrix}, \tag{2.2.29}$$

$$= \begin{pmatrix} * & * & * \\ dD - K_{r-2s}(d^{-1})^T(b^T)(a^{-1})^T K_s G & * & * \\ K_s(a^{-1})^T K_s G & \cdots & * \end{pmatrix} \begin{pmatrix} a^{-1} & \cdots & \cdots \\ 0 & \cdots & \cdots \\ 0 & 0 & \cdots \end{pmatrix}, \tag{2.2.30}$$

$$= \begin{pmatrix} * & * & * \\ dDa^{-1} - K_{r-2s}(d^{-1})^T(b^T)(a^{-1})^T K_s G a^{-1} & * & * \\ K_s(a^{-1})^T K_s G a^{-1} & \cdots & * \end{pmatrix}. \tag{2.2.31}$$

We observe the matrix:

$$\begin{pmatrix} (a^{-1})^T \otimes d & (a^{-1})^T \otimes -K_{r-2s}(d^{-1})^T(b^T)(a^{-1})^T K_s \\ 0 & (a^{-1})^T \otimes K_s(a^{-1})^T K_s \end{pmatrix}, \tag{2.2.32}$$

where $\otimes$ denotes the Kronecker-product of matrices. This matrix represents $\mathrm{Ad}(M_{a,b,c,d})$ as an endomorphism on $\mathrm{Mat}((r-2s)\times s,\mathbb{C})\times \overline{\mathfrak{so}}(s,\mathbb{C})$, acting on the pair (D,G).

The determinant of $\mathrm{Ad}(M_{a,b,c,d})$ is then the product of the determinants of the diagonal blocks from (2.2.32), acting as endomorphisms on $\mathrm{Mat}((r-2s)\times s,\mathbb{C})$ and $\overline{\mathfrak{so}}(s,\mathbb{C})$ respectively.

The upper-left block of (2.2.32) has the determinant $\det((a^{-1})^T \otimes d) = \det((a^{-1})^T)^{r-2s}\det(d)^s$. The lower-right block of (2.2.32) acting on $\mathrm{Mat}(s\times s,\mathbb{C})$, a complex vector space of dimension s^2, has the determinant $\det((a^{-1})^T\otimes K_s(a^{-1})^T K_s) = \det((a^{-1})^T)^{2s}$. When this block is restricted to acting on $\overline{\mathfrak{so}}(s,\mathbb{C})$, a complex vector space of dimension $s(s-1)/2$, we have the determinant:

$$\det((a^{-1})^T)^{2s\frac{s(s-1)/2}{s^2}} = \det((a^{-1})^T)^{s-1}. \tag{2.2.33}$$

Thus, we find that $(\det\circ\overline{\tau}_{ij})_{i,j\in I}$ is the product of cocycles of $\det(F^*\otimes(F^\perp/F))$ and $\det(F^*)^{\otimes(s-1)}$. The isomorphism in (2.2.25) then follows.

For the case of $F=F^\perp$, the calculations are the same, apart from removing the middle rows and columns in (2.2.27)…(2.2.31). The isomorphism in (2.2.26) then follows. □

If ξ is a principal-$\mathbf{SO}(2,\mathbb{C})$-bundle, nontrivial isotropic subbundles of E_ξ correspond to sections σ of ξ.

We can now present an analog of Theorem 2.2.14 for special orthogonal vector bundles.

Theorem 2.2.17. *Let $r\geq 3$, and let ξ be a principal-$\mathbf{SO}(r,\mathbb{C})$-bundle. The induced vector bundle E_ξ admits the structure of a special orthogonal vector bundle (E_ξ,β,τ). We write $r=2n+1$ if r is odd, and $r=2n$ if r is even. The following are equivalent:*

(i) The bundle ξ is Ramanathan-(semi)-stable.

(ii) For all isotropic subbundles $F\neq 0$ of (E_ξ,β) of rank $s=1,\dots,n$, where $s\neq n-1$ if r is even, we have $\mu(F)(\leq)\mu(E_\xi)=0$.

Proof. Any isotropic subbundle $F\neq 0$ of E_ξ, of rank $s\neq n-1$ if r is even, corresponds to a reduction $\sigma^*\xi$ of ξ to a maximal standard parabolic subgroup P_I of $\mathbf{SO}(r,\mathbb{C})$, due to Lemma 2.1.23. Due to Lemma 2.2.16, we have:

$$\deg(\sigma^*V_{\xi/P_I}) = -\deg(F)(r-2s)-\deg(F)(s-1), \tag{2.2.34}$$

$$= -\deg(F)(r-s-1). \tag{2.2.35}$$

Since $r\geq 3$, we have $r-s-1\geq 1$, and follow that $\deg(\sigma^*V_{\xi/P_I})(\geq)0$ is equivalent to $\mu(F)(\leq)\mu(E_\xi)=0$, so the theorem follows. □

If ξ is a principal-$\mathbf{SO}(2,\mathbb{C})$-bundle, it is trivially Ramanathan-stable, and the condition *(ii)* of Theorem 2.2.17 would be vacuous.

Finally, we mention an analog of Theorem 2.2.14 and Theorem 2.2.17 for symplectic vector bundles. By proving symplectic versions of Lemma 2.2.15 and Lemma 2.2.16, we can now present the following theorem.

Theorem 2.2.18. *Let ξ be a principal-$\mathbf{Sp}(2n,\mathbb{C})$-bundle. The induced vector bundle E_ξ admits the structure of a symplectic vector bundle (E_ξ,β). The following are equivalent:*

(i) The bundle ξ is Ramanathan-(semi)-stable.

(ii) For all isotropic subbundles $F\neq 0$ of (E_ξ,β), we have $\mu(F)(\leq)\mu(E_\xi)=0$.

Proof. Analogous to that of Theorem 2.2.17. □

Theorem 2.2.17 and Theorem 2.2.18 essentially match the results in [Ram75, Remark 3.1], as a consequence of [Ram75, Lemma 3.3]. However, in the even $r = 2n$ case of Theorem 2.2.17, our result differs slightly due to the fact that in Lemma 2.1.23, not all isotropic subbundles correspond to reductions to maximal standard parabolic subgroups P_I of $\mathbf{SO}(2n, \mathbb{C})$, with the exception being in (2.1.29).

2.2.4. The cases of derived subgroups and products

Until now, we dealt with the structure groups $\mathbf{GL}(r, \mathbb{C})$, and for $r \geq 2$, also $\mathbf{SO}(r, \mathbb{C})$ and $\mathbf{Sp}(2n, \mathbb{C})$, although we did not cover the case of $\mathbf{SL}(r, \mathbb{C})$.

In this subsection, we compare the Ramanathan-(semi)-stability of principal bundles with the Ramanathan-(semi)-stability of their reductions to derived subgroups. As a special case, we get an analog of Theorem 2.2.14 for special vector bundles, since $\mathbf{SL}(r, \mathbb{C})$ is the derived subgroup of $\mathbf{GL}(r, \mathbb{C})$.

Let G be a connected complex reductive group, with neutral element $e \in G$ and reductive Lie algebra $\mathfrak{g}$.

DEFINITION 2.2.19. The *derived subgroup* G_{der} of G is the subgroup generated by the commutators $g^{-1}h^{-1}gh = [g, h] \in [G, G]$ of elements in G.

The group G_{der} is a normal subgroup of G by construction. As shown in [Bor91, I.2.4], it is also a closed connected complex algebraic subgroup of G.

If H_1 and H_2 are subgroups of G, then H_1H_2 denotes the subgroup of G generated by elements of H_1 and H_2.

REMARK 2.2.20. (a) In [Bor91, IV.14.2 Proposition], it is proven that $G = G_{der}\mathbf{R}(G)$, and that $G_{der} \cap \mathbf{R}(G)$ is finite. Since $\mathbf{R}(G_{der})$ is contained within $G_{der} \cap \mathbf{R}(G)$, and is connected, we have that $\mathbf{R}(G_{der}) = e$, and thus G_{der} is semisimple.

(b) Since $\mathbf{R}(G) = \mathbf{Z}(G)^0$, the Lie algebra of $\mathbf{R}(G)$ is $\mathfrak{z}(\mathfrak{g})$, and the Lie algebra of G_{der} is $\mathfrak{g}_{ss}$, using the decomposition $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{z}(\mathfrak{g})$ from Theorem 1.1.5.

(c) We have the maps:

$$\{\text{Parabolic subgroups of } G\} \leftrightarrow \{\text{Parabolic subgroups of } G_{der}\}, \tag{2.2.36}$$

$$P \mapsto P_{der} = G_{der} \cap P, \tag{2.2.37}$$

$$P_{der}\mathbf{R}(G) \leftarrow P_{der}, \tag{2.2.38}$$

which are well-defined, since G/P is a complex complete algebraic variety, if and only if G_{der}/P_{der} is such, due to the isomorphism of complex algebraic varieties:

$$G/P = G_{der}P/P \simeq G_{der}/(G_{der} \cap P) = G_{der}/P_{der}. \tag{2.2.39}$$

The maps in (2.2.37) and (2.2.38) are inverse to each other, using the identification between root space decompositions of $\mathfrak{g}$ and $\mathfrak{g}_{ss}$ from (d) of Remark 1.1.11, and the correspondence between standard parabolic subgroups and subsets of simple roots from (b) of Remark 1.1.25.

Given how closely related the groups G and G_{der} are, it is natural to ask how the Ramanathan-(semi)-stability of principial-G_{der}-bundles and principal-G-bundles are related.

LEMMA 2.2.21. *A principal-G_{der}-bundle ξ is Ramanathan-(semi)-stable if and only if its extension $\xi(G)$ of ξ, with respect to the inclusion $G_{der} \hookrightarrow G$, is Ramanathan-(semi)-stable as a principal-G-bundle.*

PROOF. Let P_{der} be a maximal parabolic subgroup of G_{der}, and let $P = P_{der}\mathbf{R}(G)$ be the corresponding maximal parabolic subgroup of G. Let $\sigma_{der}^*\xi$ be a reduction of ξ to

P_{der} defined by a section $\sigma_{der} : X \to \xi/P_{der}$. Since the isomorphism in (2.2.39) is G_{der}-equivariant, we induce an isomorphism $\xi/P_{der} \simeq \xi(G)/P$ of fiber bundles, and thus the section $\sigma_{der} : X \to \xi/P_{der}$ corresponds to a section $\sigma : X \to \xi(G)/P$. In this situation, $\sigma^*(\xi(G))$ is isomorphic to an extension of $\sigma^*_{der}\xi$ to P, and both share cocycles $(\sigma_{ij})_{i,j\in I}$ that map into P_{der}.

Using that $\mathfrak{p}_{der} = \mathfrak{g}_{ss} \cap \mathfrak{p}$, and that $\mathfrak{p} = \mathfrak{p}_{der} \oplus \mathfrak{z}(\mathfrak{g})$ as Lie algebras, we claim that for all $i, j \in I$, the following diagram commutes:

$$\begin{array}{ccccccc} U_i \cap U_j & \xrightarrow{\sigma_{ij}} & P_{der} & \xrightarrow{\mathrm{Ad}} & \mathbf{GL}(\mathfrak{p}_{der}) & \xrightarrow{\det} & \mathbb{C}^\times \\ & \searrow^{\sigma_{ij}} & \downarrow & & & & \Vert \\ & & P & \xrightarrow{\mathrm{Ad}} & \mathbf{GL}(\mathfrak{p}) & \xrightarrow{\det} & \mathbb{C}^\times \end{array} \quad . \tag{2.2.40}$$

Since the adjoint representation of P_{der} acts on $\mathfrak{z}(\mathfrak{g})$ through the identity, the morphisms on the bottom row of (2.2.40) have the same determinants as the top row of (2.2.40). Hence, the diagram commutes.

As the top row of (2.2.40) gives us cocycles of $\det(\mathrm{ad}(\sigma^*_{der}\xi))$ and as the bottom row of (2.2.40) gives us cocycles of $\det(\mathrm{ad}(\sigma^*(\xi(G))))$, these two line bundles are isomorphic to each other, and thus $\deg(\mathrm{ad}(\sigma^*_{der}\xi)) = \deg(\mathrm{ad}(\sigma^*(\xi(G))))$.

Since Ramanathan-(semi)-stability is determined through these degrees, as seen in Lemma 2.2.9, the claim follows. □

The usefulness of Lemma 2.2.21 is clear in the $G = \mathbf{GL}(r, \mathbb{C})$ case.

Example 2.2.22. Let $r \geq 2$. For $G = \mathbf{GL}(r, \mathbb{C})$, $\mathbf{GL}(r, \mathbb{C})_{der}$ is generated by the matrices in $[\mathbf{GL}(r, \mathbb{C}), \mathbf{GL}(r, \mathbb{C})]$, which have determinant 1. Since the Lie algebra of $\mathbf{GL}(r, \mathbb{C})_{der}$ is $\mathfrak{gl}(r, \mathbb{C})_{ss} = \mathfrak{sl}(r, \mathbb{C})$, as we saw in (a) of Example 1.1.7, we have that $\mathbf{GL}(r, \mathbb{C})_{der} = \mathbf{SL}(r, \mathbb{C})$.

Due to Theorem 2.2.14, the slope-(semi)-stability of a special vector bundle (E, τ) is equivalent to the Ramanathan-(semi)-stability of $\mathrm{Fr}(E)$.

Since the frame bundle $\mathrm{Fr}(E)$ is isomorphic to an extension of the special frame bundle $\mathrm{Fr}_{\mathbf{SL}}(E, \tau)$ to $\mathbf{GL}(r, \mathbb{C})$, through the inclusion $\mathbf{SL}(r, \mathbb{C}) \hookrightarrow \mathbf{GL}(r, \mathbb{C})$, we use Lemma 2.2.21 to follow that the Ramanathan-(semi)-stability of $\mathrm{Fr}(E)$ is equivalent to the Ramanathan-(semi)-stability of $\mathrm{Fr}_{\mathbf{SL}}(E, \tau)$.

Altogether, the slope-(semi)-stability of E is equivalent to the Ramanathan-(semi)-stability of $\mathrm{Fr}_{\mathbf{SL}}(E, \tau)$, which is the special linear version of Theorem 2.2.14.

In this chapter, our last goal is to answer whether the correspondences from Theorem 2.2.14, Theorem 2.2.17 and Theorem 2.2.18 are compatible with products of structure groups.

For $m = 1, \ldots, l$, let G_m be a connected complex reductive group, then $G = G_1 \times \ldots \times G_l$ is a connected complex reductive group. For all $m = 1, \ldots, l$, we choose a Cartan subalgebra $\mathfrak{t}_m$ of the Lie algebra $\mathfrak{g}_m$ of G_m, and a choice of positive roots $\Phi(\mathfrak{g}_m, \mathfrak{t}_m)^+$, which induces the simple roots $\triangle_m$. We have a Cartan subalgebra $\mathfrak{t} = \mathfrak{t}_1 \oplus \ldots \oplus \mathfrak{t}_m$ of the Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l$ of G, which induces a natural bijection $\Phi(\mathfrak{g}, \mathfrak{t}) \simeq \Phi(\mathfrak{g}_1, \mathfrak{t}_1) \sqcup \ldots \sqcup \Phi(\mathfrak{g}_l, \mathfrak{t}_l)$. We choose the positive roots $\Phi(\mathfrak{g}, \mathfrak{t})^+ \simeq \Phi(\mathfrak{g}_1, \mathfrak{t}_1)^+ \sqcup \ldots \sqcup \Phi(\mathfrak{g}_l, \mathfrak{t}_l)^+$, inducing the simple roots $\triangle \simeq \triangle_1 \sqcup \ldots \sqcup \triangle_l$.

Lemma 2.2.23. *For all $m = 1, \ldots, l$, let ξ_m be a principal-G_m-bundle, then we have a principal-G-bundle $\xi = \xi_1 \times \ldots \times \xi_l$. The bundle ξ is Ramanathan-(semi)-stable if and only if for all $m = 1, \ldots, l$, ξ_m is Ramanathan-(semi)-stable.*

Proof. Let $\sigma : X \to \xi/P_I$ be a reduction of ξ to a maximal standard parabolic subgroup P_I of G.

Any maximal standard parabolic subgroup P_I of G corresponds to $I = \{\alpha_m\} \subseteq \triangle \simeq \triangle_1 \sqcup \ldots \sqcup \triangle_l$, where α_m corresponds to a simple root in $\triangle_m$, for $m = 1, \ldots, l$. By construction, P_I is equal to $G_1 \times \ldots \times (P_I)_m \times \ldots \times G_l$, where $(P_I)_m$, on the m-th component, is a maximal standard parabolic subgroup of G_m. Thus, $\sigma : X \to \xi/P_I$ appears as:

$$\sigma : X \to \xi/P_I = (\xi_1 \times \ldots \times \xi_l)/(G_1 \times \ldots \times (P_I)_m \times \ldots \times G_l), \tag{2.2.41}$$

$$\simeq X \times \ldots \times (\xi_m/(P_I)_m) \times \ldots \times X, \tag{2.2.42}$$

which corresponds to a section $\sigma_m : X \to \xi_m/(P_I)_m$. Therefore, for a fixed $m = 1, \ldots, l$, reductions of ξ to P_I, with $I = \{\alpha_m\}$, are in correspondence with reductions of ξ_m to $(P_I)_m$.

Furthermore, we get an isomorphism of vector bundles:

$$(\sigma^*\xi)(\mathfrak{g}/\mathfrak{p}_I) = P_I\backslash((\sigma^*\xi \times \mathfrak{g})/\mathfrak{p}_I) \simeq (P_I)_m\backslash((\sigma_m^*\xi_m \times \mathfrak{g}_m)/(\mathfrak{p}_I)_m) = (\sigma_m^*\xi_m)(\mathfrak{g}_m/(\mathfrak{p}_I)_m). \tag{2.2.43}$$

Due to this isomorphism, the two vertical tangent bundles $\sigma^* V_{\xi/P_I}$ and $\sigma_m^* V_{\xi_m/(P_I)_m}$ share the same degree.

Since this holds for all $m = 1, \ldots, l$, the claim of the lemma follows. □

REMARK 2.2.24. Due to this lemma, Theorem 2.2.14, Theorem 2.2.17, Theorem 2.2.18 and Remark 2.2.22, imply the following special cases:

(a) If for all $m = 1, \ldots, l$, we have $G_m = \mathbf{GL}(r_m, \mathbb{C})$, the bundle $\xi = \xi_1 \times \ldots \times \xi_l$ is Ramanathan-(semi)-stable if and only if for all $m = 1, \ldots, l$, the induced vector bundle E_{ξ_m} is slope-(semi)-stable.

(b) If for all $m = 1, \ldots, l$, we have $G_m = \mathbf{SO}(r_m, \mathbb{C})$, $r_m \geq 3$, the bundle $\xi = \xi_1 \times \ldots \times \xi_l$ is Ramanathan-(semi)-stable if and only if for all $m = 1, \ldots, l$, the induced vector bundle E_{ξ_m} fulfills the condition from Theorem 2.2.17.

(c) If for all $m = 1, \ldots, l$, we have $G_m = \mathbf{Sp}(2n_m, \mathbb{C})$, the bundle $\xi = \xi_1 \times \ldots \times \xi_l$ is Ramanathan-(semi)-stable if and only if for all $m = 1, \ldots, l$, the induced vector bundle E_{ξ_m} fulfills the condition from Theorem 2.2.18.

In (STEP 4) of the Examples in Section 1.2, we saw in that the Levi-factors L_I of the groups $\mathbf{GL}(r, \mathbb{C})$, and for $r \geq 2$, also $\mathbf{SL}(r, \mathbb{C})$, $\mathbf{SO}(r, \mathbb{C})$ and $\mathbf{Sp}(2n, \mathbb{C})$, are isomorphic to products of connected complex reductive groups.

Through Lemma 2.2.23, we can use this to test the Ramanathan-semistability of principal-L_I-bundles, which will be important for investigating canonical reductions later in Chapter 3.

CHAPTER 3

Canonical reductions of principal bundles

We follow [HN75] to construct Harder-Narasimhan filtrations of vector bundles, and prove their uniqueness. Afterward, we construct analogs of these filtrations for special orthogonal and symplectic vector bundles.

Using Harder-Narasimhan filtrations of adjoint bundles, we follow Atiyah and Bott in [AB82] to construct canonical reductions of principal-G-bundles, where G is a connected complex reductive group. We also cover the Biswas and Holla approach in [BH04] to constructing canonical reductions, and verify that these two approaches are equivalent.

As before, X is a compact connected Riemann surface, which is the base space for our bundles.

3.1. Harder-Narasimhan filtrations of vector bundles

3.1.1. The case of vector bundles

In order to investigate the slope-semistability properties of vector bundles, we introduce Harder-Narasimhan filtrations.

THEOREM 3.1.1. *Let $E \neq 0$ be a vector bundle of rank r. There exists a filtration of E by subbundles:*

$$\mathcal{F}_E : 0 = E_0 \subsetneq \ldots \subsetneq E_l = E, \tag{3.1.1}$$

that fulfills the following properties:

(i) The quotient bundles $F_m = E_m/E_{m-1}$, $m = 1, \ldots, l$, are slope-semistable.
(ii) The quotient bundles fulfill the slope inequalities:

$$\mu(F_1) > \ldots > \mu(F_l). \tag{3.1.2}$$

This filtration is unique amongst all filtrations of E with these properties.

This is called the *Harder-Narasimhan filtration* of E, first constructed in [HN75, Proposition 1.3.9]. In order to prove this theorem, we first discuss some results on degrees and slopes of vector bundles.

LEMMA 3.1.2. *Let:*

$$0 \to E' \to E \to E'' \to 0, \tag{3.1.3}$$

be a short exact sequence of nontrivial vector bundles. Then one of the following is true:

(i) $\mu(E') \leq \mu(E) \leq \mu(E'')$.
(ii) $\mu(E') \geq \mu(E) \geq \mu(E'')$.

In either case, the inequalities may all be strict, or otherwise may all be equalities.

PROOF. This follows from the additivity of ranks and degrees of vector bundles, as seen in Remark 2.2.3. This is proven in [Fri98, 4. Lemma 2]. □

For the following remark, we fix an embedding $X \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ into a complex projective space, realizing X as a complex algebraic variety.

REMARK 3.1.3. Let $\mathbf{O}_X$ be the complex algebraic structure sheaf of X. The categories of vector bundles and locally free-$\mathbf{O}_X$-modules are equivalent, where the latter is a full subcategory of the category of coherent-$\mathbf{O}_X$-modules:

$$\mathit{VecBun}_X \simeq \mathit{LocFree}\text{-}\mathbf{O}_X\text{-}\mathit{Mod} \subseteq \mathit{Coh}\text{-}\mathbf{O}_X\text{-}\mathit{Mod}. \tag{3.1.4}$$

Let $\varphi : E \to E'$ be a nontrivial morphism of vector bundles. Since $\mathit{Coh}\text{-}\mathbf{O}_X\text{-}\mathit{Mod}$ is an abelian category, we have kernels $\ker_{\mathit{Coh}}(\varphi)$ and images $\mathrm{im}_{\mathit{Coh}}(\varphi)$ of φ in $\mathit{Coh}\text{-}\mathbf{O}_X\text{-}\mathit{Mod}$, such that the homomorphism theorem $E/\ker_{\mathit{Coh}}(\varphi) \simeq \mathrm{im}_{\mathit{Coh}}(\varphi)$ is fulfilled.

For the kernels $\ker(\varphi)$ and images $\mathrm{im}(\varphi)$ of φ in VecBun_X, it can be shown that $\ker(\varphi) = \ker_{\mathit{Coh}}(\varphi) \neq E$, and thus $\mathrm{im}_{\mathit{Coh}}(\varphi) \neq 0$. Through an application of the Riemann-Roch formula, as seen in [Fri98, 2. Theorem 2], it can be shown that:

$$\mu(E/\ker(\varphi)) = \mu(\mathrm{im}_{\mathit{Coh}}(\varphi)) \leq \mu(\mathrm{im}(\varphi)). \tag{3.1.5}$$

LEMMA 3.1.4. *Let $E \neq 0$. We define:*

$$\mathit{deg}_{\max}(E) = \sup\{\mathit{deg}(F) | F \neq 0 \textit{ is a subbundle of } E\}, \tag{3.1.6}$$

$$\mu_{\max}(E) = \sup\{\mu(F) | F \neq 0 \textit{ is a subbundle of } E\}. \tag{3.1.7}$$

We have $\mathit{deg}_{\max}(E), \mu_{\max}(E) < \infty$, and there exists subbundles $F \neq 0$ and $F' \neq 0$ of E, such that $\mathit{deg}(F) = \mathit{deg}_{\max}(E)$ and $\mu(F') = \mu_{\max}(E)$.

PROOF. We first handle $\deg_{\max}(E)$, where it suffices to show that $\deg_{\max}(E) < \infty$, since degrees are integers.

The 0-th sheaf cohomology of a vector bundle is the complex vector space of global sections. For a subbundle $F \neq 0$ of E of rank $s = 1, \dots, r$, a global section of F is also a global section of E, and thus $\dim_{\mathbb{C}}(H^0(X, E)) \geq \dim_{\mathbb{C}}(H^0(X, F))$.

We denote the genus of X by g, we then have due to Riemann-Roch, from [Fri98, 2. Theorem 2], that:

$$\dim_{\mathbb{C}}(H^0(X, F)) = \dim_{\mathbb{C}}(H^1(X, F)) + \deg(F) + (1-g)s \geq \deg(F) + (1-g)s, \tag{3.1.8}$$

$$\dim_{\mathbb{C}}(H^0(X, E)) + (g-1)s \geq \dim_{\mathbb{C}}(H^0(X, F)) + (g-1)s \geq \deg(F). \tag{3.1.9}$$

If $g = 0$, we have $\dim_{\mathbb{C}}(H^0(X, E)) \geq \deg(F)$ due to (3.1.9). Otherwise, $g \geq 1$, and we can imply $\dim_{\mathbb{C}}(H^0(X, E)) + (g-1)r \geq \deg(F)$ using (3.1.9), so the claim for $\deg_{\max}(E)$ follows.

We now handle $\mu_{\max}(E)$. Let $s = 1, \dots, r$, then using (3.1.9), we find a subbundle F_s of E of rank s, such that F_s has maximal degree amongst all subbundles of E of rank s. Due to the finite choices of s, we follow $\mu_{\max}(E) = \max_{s=1}^{r} \mu(F_s) < \infty$, and thus there exists an s such that $\mu(F_s) = \mu_{\max}(E)$. □

We can now use $\deg_{\max}$ and $\mu_{\max}$, and slope-semistability, to investigate morphisms of vector bundles.

LEMMA 3.1.5. *Let $\varphi : E \to E'$ be a morphism of nontrivial vector bundles. If E is slope-semistable such that $\mu(E) > \mu_{\max}(E')$, then $\varphi = 0$.*

PROOF. If $\varphi \neq 0$, we can use Lemma 3.1.2 and Remark 3.1.3 to follow:

$$\mu_{\max}(E') < \mu(E) \leq \mu(E/\ker(\varphi)) \leq \mu(\mathrm{im}(\varphi)), \tag{3.1.10}$$

which is a contradiction to the maximality of $\mu_{\max}(E')$. □

The proof of Theorem 3.1.1 also requires constructing maximal destabilizing subbundles.

LEMMA 3.1.6. *For a subbundle $F \neq 0$ of E of maximal rank, such that $\mu(F) = \mu_{\max}(E)$, we have that:*

(i) The subbundle F is slope-semistable.

(ii) Either $F = E$ or $\mu_{\max}(E) > \mu_{\max}(E/F)$.

Such a subbundle F exists, and is unique as a subbundle of E.

We call F the *maximal destabilizing subbundle* $\mathcal{G}(E)$ of E.

PROOF. To prove *(i)*, let $F' \neq 0$ be a subbundle of F, then since F' is a subbundle of E, we have $\mu(F') \leq \mu_{\max}(E) = \mu(F)$, implying slope-semistability.

To prove *(ii)*, assuming $F \neq E$, let $E'/F \neq 0$ be a subbundle of E/F, such that $\mu(E'/F) = \mu_{\max}(E/F)$, which exists due to Lemma 3.1.4. Due to $\mu(F) = \mu_{\max}(E)$, and the maximality of the rank of F, we have $\mu(E') < \mu(F)$. Then by using Lemma 3.1.2 on the short exact sequence:

$$0 \to F \to E' \to E'/F \to 0, \tag{3.1.11}$$

we have $\mu_{\max}(E) = \mu(F) > \mu(E'/F) = \mu_{\max}(E/F)$.

To prove the uniqueness of F, let F' be another such subbundle of E with the same properties as F. If $F = F' = E$, there is nothing to show, otherwise we may assume $F' \neq E$ or $F \neq E$. Without loss of generality, let us assume $F \neq E$. The projection $\varphi : F' \to E/F$ fulfills $\mu(F') = \mu_{\max}(E) > \mu_{\max}(E/F)$, and F' is slope-semistable. Using Lemma 3.1.5, we have $\varphi = 0$, and thus F' is a subbundle of F. Due to the maximality of F' and F with respect to rank, we get $F' = F$. □

In order to prove Theorem 3.1.1, we also need that the conditions *(i)* and *(ii)* of Lemma 3.1.6 characterize maximal destabilizing subbundles.

LEMMA 3.1.7. *For a slope-semistable subbundle $F \neq 0$ of E, such that $F = E$ or $\mu_{\max}(E) > \mu_{\max}(E/F)$, we have $F = \mathcal{G}(E)$.*

PROOF. If $F = E$, then E is slope-semistable, and $F = E = \mathcal{G}(E)$ is the maximal destabilizing subbundle of E.

Otherwise, $F \neq E$. The projection $\varphi : \mathcal{G}(E) \to E/F$ fulfills $\mu(\mathcal{G}(E)) = \mu_{\max}(E) > \mu_{\max}(E/F)$, and thus $\varphi = 0$, due to Lemma 3.1.5. Thus, $\mathcal{G}(E)$ is a subbundle of F. The slope-semistability of F implies that $\mu(\mathcal{G}(E)) = \mu_{\max}(E) = \mu(F)$, and thus $\mathcal{G}(E) = F$, using the maximality of the rank of $\mathcal{G}(E)$. □

We can now finally prove the existence and uniqueness of the Harder-Narasimhan filtration.

PROOF OF THEOREM 3.1.1. We construct a Harder-Narasimhan filtration of E starting with $E_0 = 0$ and $E_1 = \mathcal{G}(E)$. If $E_1 \neq E$, we construct E_2 as a subbundle of E containing E_1, such that $E_2/E_1 = \mathcal{G}(E/E_1)$. In general, for increasing $m \in \mathbb{N}$, we construct E_m recursively such that $E_m/E_{m-1} = \mathcal{G}(E/E_{m-1})$, which terminates at $E_l = E$ for some $l \in \mathbb{N}$, due to the rank r of E being finite.

For all $m = 1, \ldots, l$, since $F_m = E_m/E_{m-1}$ a maximal destabilizing subbundle of E/E_{m-1}, it is slope-semistable due to Lemma 3.1.6.

If $l = 1$, we have already found a Harder-Narasimhan filtration, namely:

$$\mathcal{F}_E : 0 = E_0 \subsetneq E_1 = E. \tag{3.1.12}$$

Otherwise, we assume $l \geq 2$, where it remains to verify the slope inequalities on the quotient bundles. For all $m = 1, \ldots, l-1$, we have $F_m \neq E/E_{m-1}$ and $(E/E_{m-1})/F_m \simeq E/E_m$, thus we have due to Lemma 3.1.6:

$$\mu(F_m) = \mu_{\max}(E/E_{m-1}) > \mu_{\max}(E/E_m) \geq \mu(F_{m+1}), \tag{3.1.13}$$

and we have thus found a Harder-Narasimhan filtration $\mathcal{F}_E$ of E.

We now prove the uniqueness of Harder-Narasimhan filtrations. Let the following filtration:

$$\overline{\mathcal{F}}_E : 0 = E_0 \subsetneq \ldots \subsetneq E_l = E, \tag{3.1.14}$$

denote any Harder-Narasimhan filtration, with the quotient bundles $F_m = E_m/E_{m-1}$, $m = 1, \dots, l$. We claim that for all $m = 1, \dots, l$, we have $\mathcal{G}(E/E_{m-1}) = F_m$, and thus this filtration coincides with the filtration $\mathcal{F}_E$ we constructed earlier.

We perform an induction on l. If $l = 1$, the statement is clear, since $\mathcal{G}(E/E_0) = \mathcal{G}(E) = E = F_1$, as E is slope-semistable.

Assuming the statement is true for $l - 1 \geq 1$. For $m = 2, \dots, l$, we have that $(E_m/E_1)/(E_{m-1}/E_1) \simeq F_m$, which is slope-semistable, and thus E/E_1 has a Harder-Narasimhan filtration:

$$\overline{\mathcal{F}}_{E/E_1} : 0 = E_1/E_1 \subsetneq \dots \subsetneq E_l/E_1 = E/E_1, \tag{3.1.15}$$

of length $l - 1$, which is thus unique.

Let $m = 2, \dots, l$. From the induction hypothesis, we have $\mathcal{G}((E/E_1)/(E_{m-1}/E_1)) = (E_m/E_1)/(E_{m-1}/E_1)$. Through the isomorphism $(E/E_1)/(E_{m-1}/E_1) \simeq (E/E_{m-1})$, which restricts to $(E_m/E_1)/(E_{m-1}/E_1) \simeq F_m$, we follow that $\mathcal{G}(E/E_{m-1}) = F_m$.

It remains to show that $\mathcal{G}(E/E_0) = F_1$, i.e., $\mathcal{G}(E) = E_1$. We have that $E_1 = F_1$ is slope-semistable, and since $l - 1 \geq 1$, we have $E_1 \neq E$. Due to Lemma 3.1.7, it suffices to show that $\mu_{\max}(E) > \mu_{\max}(E/E_1)$.

Since $\mathcal{G}(E/E_1) = F_2$, we have $\mu(F_2) = \mu_{\max}(E/E_1)$. Using $\mu(F_1) > \mu(F_2)$, we follow that $\mu_{\max}(E) > \mu_{\max}(E/E_1)$. Thus, we have $\mathcal{G}(E) = E_1$.

In conclusion, Harder-Narasimhan filtrations exist and are unique. □

As an example, we explain how Harder-Narasimhan filtrations appear for direct sums $E = L_1 \oplus \dots \oplus L_l$ of line bundles. For this, we need the following remark and lemma.

REMARK 3.1.8. Let E be a vector bundle of rank r, and let F and F' be subbundles of E. Despite the sets $F \cap F'$ and $F + F'$ having complex vector spaces as their fibers, they may not be vector bundles, as they may not be locally trivial. To resolve this, we define the morphisms $\varphi : F \oplus F' \to E$ and $\psi : E \to (E/F) \oplus (E/F')$:

$$\varphi : (v, w) \mapsto v + w, \quad \psi : v \mapsto ([v], [v]), \tag{3.1.16}$$

and define $F \vee F' = \mathrm{im}(\varphi)$ as the *sum* of F and F', and $F \wedge F' = \ker(\psi)$ as the *intersection* of F and F', both of which are subbundles of E. Due to Remark 2.1.3, we have $F + F' \subseteq F \vee F'$ and $F \wedge F' \subseteq F \cap F'$ as sets.

The projection $\varphi : F \to E/F'$ has the kernel $\ker(\varphi) = F \wedge F'$ and image $\mathrm{im}(\varphi) = (F \vee F')/F$. If φ is nontrivial, then Remark 3.1.3 gives us the inequality $\mu(F/(F \wedge F')) \leq \mu((F \vee F')/F')$.

We can now prove that direct sums of slope-semistable bundles of the same slope are slope-semistable.

LEMMA 3.1.9. *Let:*

$$0 \to E' \xrightarrow{\varphi} E \xrightarrow{\psi} E'' \to 0, \tag{3.1.17}$$

be a short exact sequence of nontrivial vector bundles, such that E' and E'' are slope-semistable and $\mu(E') = \mu(E'')$. Then E is semistable such that $\mu(E) = \mu(E') = \mu(E'')$.

PROOF. By applying Lemma 3.1.2 on the short exact sequence in (3.1.17), we have $\mu(E) = \mu(E') = \mu(E'')$.

We now prove the slope-semistability of E. To calculate the slope of $\mathrm{im}(\varphi) = \ker(\psi)$, we use that $\mu(E/\ker(\psi)) \leq \mu(\mathrm{im}(\psi)) = \mu(E'') = \mu(E)$, from Remark 3.1.3, and the short exact sequence:

$$0 \to \ker(\psi) \to E \to E/\ker(\psi) \to 0, \tag{3.1.18}$$

to imply $\mu(\mathrm{im}(\varphi)) = \mu(\ker(\psi)) \leq \mu(E)$ using Lemma 3.1.2. Since $\mathrm{im}_{Coh}(\varphi) \simeq E'$, from Remark 3.1.3, we also follow $\mu(E) = \mu(\mathrm{im}_{Coh}(\varphi)) \leq \mu(\mathrm{im}(\varphi))$, implying:

$$\mu(\mathrm{im}(\varphi)) = \mu(E) = \mu(\mathrm{im}_{Coh}(\varphi)) = \mu(E/\ker(\psi)). \tag{3.1.19}$$

Using Riemann-Roch, we follow:

$$E' \simeq \mathrm{im}_{Coh}(\varphi) = \mathrm{im}(\varphi) = \ker(\psi), \quad E/\ker(\psi) \simeq \mathrm{im}_{Coh}(\psi) = \mathrm{im}(\psi) = E''. \tag{3.1.20}$$

Let $F \neq 0$ be a subbundle of E, we claim that $\mu(F) \leq \mu(E)$. If F is a subbundle of $\mathrm{im}(\varphi)$, the claim follows directly due to the semistability of E'. Otherwise, F is not a subbundle of $\mathrm{im}(\varphi) = \ker(\psi)$, and thus $F/(F \wedge \ker(\psi)) \neq 0$, and $(F \vee \ker(\psi))/\ker(\psi) \neq 0$ is isomorphic to a subbundle of E''. By using Remark 3.1.8, we have:

$$\mu(F/(F \wedge \ker(\psi))) \leq \mu((F \vee \ker(\psi))/\ker(\psi)) \leq \mu(E'') = \mu(E). \tag{3.1.21}$$

If $F \wedge \ker(\psi) = 0$, then we have $\mu(F) \leq \mu(E)$. Otherwise, $F \wedge \ker(\psi) \neq 0$ is a nontrivial subbundle of $\ker(\psi)$, and $\mu(F \wedge \ker(\psi)) \leq \mu(E') = \mu(E)$, thus by using Lemma 3.1.2, the short exact sequence:

$$0 \to F \wedge \ker(\psi) \to F \to F/(F \wedge \ker(\psi)) \to 0, \tag{3.1.22}$$

implies $\mu(F) \leq \mu(E)$. □

A consequence of this lemma is that direct sums of slope-semistable bundles of the same slope are slope-semistable. We use this to describe Harder-Narasimhan filtrations of direct sums $E = L_1 \oplus \ldots \oplus L_l$ of line bundles.

Example 3.1.10. Let $E = L_1 \oplus \ldots \oplus L_l$ be the direct sum of line bundles, where we order the slopes:

$$\mu(L_1) = \ldots = \mu(L_{t_1}) > \mu(L_{t_1+1}) = \ldots = \mu(L_{t_2}) > \ldots > \mu(L_{t_k+1}) = \ldots = \mu(L_l). \tag{3.1.23}$$

Using Lemma 3.1.9, we see that the filtration:

$$\mathcal{F}_E : 0 \subsetneq \bigoplus_{i=1}^{t_1} L_i \subsetneq \bigoplus_{i=1}^{t_2} L_i \subsetneq \ldots \subsetneq \bigoplus_{i=1}^{l} L_i = E, \tag{3.1.24}$$

is the Harder-Narasimhan filtration of E.

Such a bundle E appears naturally. For example, if $X = \mathbb{P}^1_{\mathbb{C}}$ is the projective curve, then the Grothendieck splitting theorem guarantees that all vector bundles E on X are isomorphic to a direct sum of line bundles.

3.1.2. The cases of orthogonal and symplectic vector bundles

Having proven that Harder-Narasimhan filtrations for vector bundles exist and are unique, we ask whether similar filtrations exist for special orthogonal vector bundles and symplectic vector bundles.

Let (E, β, τ) be a special orthogonal vector bundle of rank $r \geq 2$.

Theorem 3.1.11. *There exists a filtration of (E, β) by isotropic subbundles of E:*

$$\mathcal{F}_{(E,\beta)} : 0 = E_0 \subsetneq \ldots \subsetneq E_l \subsetneq E, \tag{3.1.25}$$

such that the following properties hold:

(i) The quotient bundles $F_m = E_m/E_{m-1}$, $m = 1, \ldots, l$, are slope-semistable.
(ii) The quotient bundles fulfill the slope inequalities:

$$\mu(F_1) > \ldots > \mu(F_l) > 0, \tag{3.1.26}$$

(iii) Either $E_l = E_l^{\perp}$, or the rank of $E_l^{\perp}/E_l$ is $r' \geq 3$, and $Fr_{\mathbf{SO}}(E_l^{\perp}/E_l, \beta_{E_l}, \tau_{E_l})$ is Ramanathan-semistable.

This filtration is unique amongst all filtrations of (E, β, τ) with these properties.

This is called a *special orthogonal Harder-Narasimhan filtration* of (E, β, τ).

Note that condition *(iii)* can be characterized using isotropic subbundles of $(E_l^\perp/E_l, \beta_{E_l})$, using Theorem 2.2.17.

The proof approach is the same as that of Harder-Narasimhan filtrations from Theorem 3.1.1, where we need special orthogonal versions of Lemma 3.1.4, Lemma 3.1.6 and Lemma 3.1.7.

LEMMA 3.1.12. *We define:*

$$deg^{\mathrm{orth}}_{\max}(E, \beta) = \sup\{deg(F) | F \neq 0 \text{ is an isotropic subbundle of } (E, \beta)\}, \tag{3.1.27}$$

$$\mu^{\mathrm{orth}}_{\max}(E, \beta) = \sup\{\mu(F) | F \neq 0 \text{ is an isotropic subbundle of } (E, \beta)\}. \tag{3.1.28}$$

If E has nontrivial isotropic subbundles, we have $deg^{\mathrm{orth}}_{\max}(E, \beta), \mu^{\mathrm{orth}}_{\max}(E, \beta) < \infty$, and there exists isotropic subbundles $F \neq 0$ and $F' \neq 0$ of (E, β), such that $deg(F) = deg^{\mathrm{orth}}_{\max}(E, \beta)$ and $\mu(F') = \mu^{\mathrm{orth}}_{\max}(E, \beta)$.

PROOF. The case of $\deg(F) = \deg^{\mathrm{orth}}_{\max}(E, \beta) < \infty$ directly follows from $\deg^{\mathrm{orth}}_{\max}(E, \beta) \leq \deg_{\max}(E) < \infty$, using Lemma 3.1.4.

We now handle $\mu^{\mathrm{orth}}_{\max}(E, \beta)$. Let $s = 1, \ldots, n$, using (3.1.9) from Lemma 3.1.4, we find an isotropic subbundle F_s of (E, β) of rank s, such that F_s has maximal degree amongst all isotropic subbundles of (E, β) of rank s. Due to the finite choices of s, we follow $\mu^{\mathrm{orth}}_{\max}(E, \beta) = \max_{s=1}^r \mu(F_s) < \infty$, and thus there exists an s, such that $\mu(F_s) = \mu^{\mathrm{orth}}_{\max}(E, \beta)$. □

LEMMA 3.1.13. *For an isotropic subbundle $F \neq 0$ of (E, β) of maximal rank, such that $\mu(F) = \mu^{\mathrm{orth}}_{\max}(E, \beta)$, we have that:*

- *(i) The bundle F is slope-semistable.*
- *(ii) The bundle F is maximally isotropic, i.e., isotropic of maximal rank, or otherwise $\mu^{\mathrm{orth}}_{\max}(E, \beta) > \mu^{\mathrm{orth}}_{\max}(F^\perp/F, \beta_F)$.*

If $\mu(F) > 0$, then F is unique with these properties.

With the condition $\mu(F) > 0$, we call F the *maximal destabilizing isotropic subbundle* $\mathcal{G}^{\mathrm{orth}}(E, \beta)$ of (E, β).

PROOF. To prove *(i)*, let $F' \neq 0$ be a subbundle of F, then since F' is an isotropic subbundle of (E, β), we have $\mu(F') \leq \mu^{\mathrm{orth}}_{\max}(E, \beta) = \mu(F)$, implying slope-semistability.

To prove *(ii)*, assuming F is not maximally isotropic, let $F'/F \neq 0$ be an isotropic subbundle of $(F^\perp/F, \beta_F)$, such that $\mu(F'/F) = \mu^{\mathrm{orth}}_{\max}(F^\perp/F, \beta_F)$, which exists due to Lemma 3.1.12. Due to $\mu(F) = \mu^{\mathrm{orth}}_{\max}(E, \beta)$, and the maximality of the rank of F, we have $\mu(F') < \mu(F)$. Using Lemma 3.1.2 on the short exact sequence:

$$0 \to F \to F' \to F'/F \to 0, \tag{3.1.29}$$

we have $\mu^{\mathrm{orth}}_{\max}(E, \beta) = \mu(F) > \mu(F'/F) = \mu^{\mathrm{orth}}_{\max}(F^\perp/F, \beta_F)$.

To prove the uniqueness of F, given $\mu(F) > 0$, we let F' be another isotropic subbundle with the same properties as F, and claim that $F = F'$.

We perform an induction on the rank r of E. For $r = 2$, the statement is clear. Assuming the statements are true for ranks $2, \ldots, r-1$, we prove the statement for r. We first claim that $F \wedge F' \neq 0$, otherwise, $(F \vee F', \beta_{F \vee F'}, \tau_{F \vee F'})$ is a special orthogonal vector bundle admitting the nontrivial epimorphism:

$$\varphi : F \oplus F' \to F \vee F', \quad (v, w) \mapsto v + w. \tag{3.1.30}$$

Due to Lemma 3.1.9, $F \oplus F'$ is slope-semistable and:

$$2\mu(F) = \mu(F \oplus F') > \mu^{\mathrm{orth}}_{\max}(F \vee F', \beta_{F \vee F'}) = \mu(F), \tag{3.1.31}$$

implying $\varphi = 0$ using Lemma 3.1.5, leading to a contradiction.

Since $F \wedge F' \neq 0$, we can project F and F' into the special orthogonal vector bundle $((F \vee F')/(F \wedge F'), \beta_{F \vee F'}, \tau_{F \vee F'})$. Since this bundle is of lower rank than E, we can use the induction hypothesis to follow that $F = F'$. $\square$

LEMMA 3.1.14. *For a slope-semistable isotropic subbundle $F \neq 0$ of (E, β), with slope $\mu(F) > 0$, that is maximally isotropic or $\mu^{\text{orth}}_{\max}(E, \beta) > \mu^{\text{orth}}_{\max}(F^\perp/F, \beta_F)$, we have $F = \mathcal{G}^{\text{orth}}(E, \beta)$.*

PROOF. We perform an induction on the rank r of E. For $r = 2$, the statement is clear. Assuming the statements are true for ranks $2, \ldots, r-1$, we prove the statement for r. We first claim that $F \wedge \mathcal{G}^{\text{orth}}(E, \beta) \neq 0$, which follows analogously to the proof of uniqueness in Lemma 3.1.13.

Since $F \wedge \mathcal{G}^{\text{orth}}(E, \beta) \neq 0$, we can project F and $\mathcal{G}^{\text{orth}}(E, \beta)$ into the special orthogonal vector bundle $((F \vee \mathcal{G}^{\text{orth}}(E, \beta))/(F \wedge \mathcal{G}^{\text{orth}}(E, \beta)), \beta_{F \vee \mathcal{G}^{\text{orth}}(E,\beta)}, \tau_{F \vee \mathcal{G}^{\text{orth}}(E,\beta)})$. Since this bundle is of lower rank than E, we can use the induction hypothesis to follow that $F = \mathcal{G}^{\text{orth}}(E, \beta)$. $\square$

Using these results, we can now prove of Theorem 3.1.11 with a similar approach to Theorem 3.1.1.

PROOF OF THEOREM 3.1.11. We construct a special orthogonal Harder-Narasimhan filtration of (E, β) starting with $E_0 = 0$. If E is isomorphic to the product bundle $X \times \mathbb{C}^2$ of rank 2, we have finished constructing the filtration at $l = 0$. Likewise, if the rank of E is $r \geq 3$ and $\text{Fr}_{\mathbf{SO}}(E, \beta, \tau)$ is Ramanathan-semistable, we have also finished constructing the filtration at $l = 0$. Otherwise, we set $E_1 = \mathcal{G}^{\text{orth}}(E, \beta)$.

In general, for increasing $m \in \mathbb{N}$, if E_{m-1} fulfills the stopping condition *(iii)* of this Theorem, we have finished constructing the filtration at $l = m - 1$. Otherwise, we construct E_m recursively such that $E_m/E_{m-1} = \mathcal{G}^{\text{orth}}(E^\perp_{m-1}/E_{m-1}, \beta_{E_{m-1}})$, which terminates at E_l for some $l \in \mathbb{N}$.

For all $m = 1, \ldots, l$, we have that $F_m = E_m/E_{m-1}$ is slope-semistable, as a maximal destabilizing isotropic subbundle of $(E^\perp_{m-1}/E_{m-1}, \beta_{E_{m-1}})$.

If $l = 1$, we have already found a Harder-Narasimhan filtration, namely:

$$\mathcal{F}_{(E,\beta)} : 0 = E_0 \subsetneq E_1 \subsetneq E. \tag{3.1.32}$$

Otherwise, we assume $l \geq 2$, where it remains to verify the slope inequalities on the quotient bundles. For all $m = 1, \ldots, l-1$, we have that F_m is not maximally isotropic in $(E^\perp_{m-1}/E_{m-1}, \beta_{E_{m-1}})$ and $F^\perp_m/F_m \simeq E^\perp_m/E_m$, thus we have due to Lemma 3.1.13:

$$\mu(F_m) = \mu^{\text{orth}}_{\max}(E^\perp_{m-1}/E_{m-1}, \beta_{E_{m-1}}) > \mu^{\text{orth}}_{\max}(E^\perp_m/E_m, \beta_{E_m}) = \mu(F_{m+1}) > 0. \tag{3.1.33}$$

Altogether, we have proven the existence of a special orthogonal Harder-Narasimhan filtration:

$$\mathcal{F}_{(E,\beta)} : 0 = E_0 \subsetneq \ldots \subsetneq E_l \subsetneq E. \tag{3.1.34}$$

The uniqueness of $\mathcal{F}_{(E,\beta)}$ is proved analogously to the proof of Theorem 3.1.1 by performing an induction on l. $\square$

We now mention a symplectic version.

THEOREM 3.1.15. *Let (E, β) be a symplectic vector bundle of rank $r = 2n$. There exists a filtration of (E, β) by isotropic subbundles of E:*

$$\mathcal{F}_{(E,\beta)} : 0 = E_0 \subsetneq \ldots \subsetneq E_l \subsetneq E, \tag{3.1.35}$$

such that the following properties hold:

(i) The quotient bundles $F_m = E_m/E_{m-1}$, $m = 1, \ldots, l$, are slope-semistable.

(ii) The quotient bundles fulfill the slope inequalities:

$$\mu(F_1) > \ldots > \mu(F_l) > 0. \tag{3.1.36}$$

(iii) *Either* $E_l = E_l^\perp$, *or the rank of* $E_l^\perp/E_l$ *is* $r' \geq 2$, *and* $Fr_{\mathbf{Sp}}(E_l^\perp/E_l, \beta_{E_l})$ *is Ramanathan-semistable.*

This filtration is unique amongst all filtrations of (E, β) *with these properties.*

This is called the *symplectic Harder-Narasimhan filtration* of (E, β).

PROOF. Analogous to that of Theorem 3.1.11. □

3.2. Canonical reductions of principal bundles

Lemma 2.1.15, Lemma 2.1.23 and Lemma 2.1.26 show that, generally speaking, filtrations of vector bundles correspond to reductions of principal bundles to standard parabolic subgroups. Having constructed Harder-Narasimhan filtrations, we want to determine which reductions of principal bundles to standard parabolic subgroups correspond to Harder-Narasimhan filtrations.

In the general setting where G is a connected complex reductive group, these reductions are called canonical reductions of principal-G-bundles. These were first introduced by Atiyah and Bott in [AB82, Chapter 10] using adjoint bundles. Later on, Biswas and Holla in [BH04] found a characterization of canonical reductions using simple roots and dominant characters.

3.2.1. The Atiyah-Bott approach

Let ξ be a principal-G-bundle. This approach in [AB82, Chapter 10] involves finding a filtration $\mathcal{F}$ of the adjoint bundle $\mathrm{ad}(\xi)$, such that a subbundle in $\mathcal{F}$ is a parabolic Lie algebra subbundle of $\mathrm{ad}(\xi)$.

We know that the adjoint bundle is an orthogonal vector bundle, as seen in (b) of Remark 2.2.7. The following lemma proves that it is also a special orthogonal vector bundle.

LEMMA 3.2.1. *The adjoint bundle* $ad(\xi)$ *is a special orthogonal vector bundle.*

PROOF. Due to (b) of Remark 2.2.7, it suffices to show that $\det(\mathrm{ad}(\xi))$ is isomorphic to the product bundle $X \times \mathbb{C}$. As a linear algebraic group, G can be viewed as a matrix subgroup of $\mathbf{GL}(r, \mathbb{C})$, such that the adjoint representation $\mathrm{Ad} : G \to \mathbf{GL}(\mathfrak{g})$ can be viewed as matrix conjugation.

Cocycles $(\sigma_{ij})_{i,j\in I}$ of ξ induce cocycles $(\mathrm{Ad} \circ \sigma_{ij})_{i,j\in I}$ of $\mathrm{ad}(\xi)$, which are equal to $((\sigma_{ij}^{-1})^T \otimes \sigma_{ij})_{i,j\in I}$. This induces cocycles $(\det((\sigma_{ij}^{-1})^T \otimes \sigma_{ij}))_{i,j\in I}$ of $\det(\mathrm{ad}(\xi))$, which are cocycles of the product bundle $X \times \mathbb{C}$. Thus, the claim follows. □

We fix β and τ such that $(\mathrm{ad}(\xi), \beta, \tau)$ is a special orthogonal vector bundle. We have the special orthogonal Harder-Narasimhan filtration from Theorem 3.1.11:

$$\mathcal{F}_{(\mathrm{ad}(\xi),\beta)} : 0 = \mathrm{ad}(\xi)_0 \subsetneq \ldots \subsetneq \mathrm{ad}(\xi)_l \subsetneq \mathrm{ad}(\xi). \tag{3.2.1}$$

By including the corresponding coisotropic subbundles, we can extend the filtration:

$$0 = \mathrm{ad}(\xi)_0 \subsetneq \ldots \subsetneq \mathrm{ad}(\xi)_l \subseteq \mathrm{ad}(\xi)_l^\perp \subsetneq \ldots \subsetneq \mathrm{ad}(\xi)_0^\perp = \mathrm{ad}(\xi). \tag{3.2.2}$$

We claim that $\mathrm{ad}(\xi)_l^\perp$ is a parabolic Lie subbundle of $\mathrm{ad}(\xi)$. To prove this, we first need the following results on slope-stability.

LEMMA 3.2.2. *Let* $E \neq 0$ *and* $E' \neq 0$ *be vector bundles of rank* r *and* r' *respectively. If* E *and* E' *are slope-semistable, the bundle* $E \otimes E'$ *is slope-semistable of rank* rr'*, with slope* $\mu(E \otimes E') = \mu(E) + \mu(E')$.

PROOF. Due to properties of tensor products, it is clear that $E \otimes E'$ has rank rr'. Using properties of the degree on tensor bundles, we obtain:

$$\mu(E \otimes E') = \deg(E \otimes E')/rr' = (r'\deg(E) + r\deg(E))/rr' = \mu(E) + \mu(E'). \tag{3.2.3}$$

The statement about slope-semistability is nontrivial. A proof is sketched in [AB82, Lemma 10.1], referencing the papers [NS65] and [Don83], that prove the case when E and E' are slope-stable.

In [NS65, Corollary 2], Narasimhan and Seshadri concluded that slope-stable vector bundles, of degree 0, are isomorphic to associated bundles of irreducible unitary representations of the fundamental group of X. As the tensor product of two unitary representations is again a unitary representation, the slope-semistability of $E \otimes E'$ follows from [NS65, Proposition 10.4].

Donaldson in [Don83, Theorem] provides another proof, when E and E' are slope-stable, using unitary connections of vector bundles. This has the advantage that the degree of E and E' no longer has to be 0.

Following [AB82, Lemma 10.1], to generalize to the situation where E and E' are only slope-semistable, we may use Jordan-Hölder filtrations of E and E'. ☐

LEMMA 3.2.3. *Let $E \neq 0$ and $E' \neq 0$ be vector bundles with filtrations by subbundles:*

$$0 = E_0 \subsetneq \ldots \subsetneq E_l = E, \quad 0 = E'_0 \subsetneq \ldots \subsetneq E'_{l'} = E', \tag{3.2.4}$$

with slope-semistable quotients $F_m = E_m/E_{m-1}$, $m = 1, \ldots l$, and $F'_m = E'_m/E'_{m-1}$, $m = 1, \ldots l'$.

Let $q \in \mathbb{Q}$. Assuming the quotients fulfill $\mu(F_m) \geq q$, and $\mu(F'_m) < q$, every morphism $E \to E'$ of vector bundles is 0.

PROOF. If $l = l' = 1$, then E and E' are slope-semistable, for which the claim follows from Lemma 3.1.5.

We perform a double induction on (l, l'). Assuming the claim is true for the pairs of lengths $(l,\, l')$, we wish to verify the claim for the lengths $(l+1,\, l')$, and $(l,\, l'+1)$. For $(l+1,\, l')$, the restriction of a morphism $E \to E'$ to $E_l \to E'$ is 0 due to the induction hypothesis. Similarly, $E/E_l \to E'$ is also 0 due to the induction hypothesis. Altogether, we have that $E \to E'$ is 0. The argument for $(l,\, l'+1)$ is analogous to of $(l+1,\, l')$. Thus, the claim of the lemma follows. ☐

LEMMA 3.2.4. *For the filtration (3.2.2) of the adjoint bundle ad(ξ):*

(i) For $m = 0, \ldots, l$, the bundle ad$(\xi)_m$ is a nilpotent Lie algebra bundle.

(ii) The bundle ad$(\xi)_l^\perp$ is a parabolic Lie algebra subbundle of ad(ξ).

PROOF. We first prove that $\mathrm{ad}(\xi)_l^\perp$ is a Lie algebra bundle. It suffices to show that the map $\varphi : \mathrm{ad}(\xi)_l^\perp \otimes \mathrm{ad}(\xi)_l^\perp \to \mathrm{ad}(\xi)/\mathrm{ad}(\xi)_l^\perp$ induced by the Lie bracket is 0. Using Lemma 3.2.2, there exists a filtration of $\mathrm{ad}(\xi)_l^\perp \otimes \mathrm{ad}(\xi)_l^\perp$, whose quotients are semistable of slope greater or equal to 0. Furthermore, the slopes of the quotients of the Harder-Narasimhan filtration of $\mathrm{ad}(\xi)_l/\mathrm{ad}(\xi)_l^\perp$ are lesser than 0. Using Lemma 3.2.3, we have that $\varphi = 0$.

For *(i)*, a similar argument shows that for $m = 1, \ldots, l$, the map $\varphi : \mathrm{ad}(\xi)_l \otimes \mathrm{ad}(\xi)_m \to \mathrm{ad}(\xi)/\mathrm{ad}(\xi)_{m-1}$ induced by the Lie bracket is 0.

For *(ii)*, we note that for all $x \in X$, we have $(\mathrm{ad}(\xi)_l^\perp)_x \simeq (\mathrm{ad}(\xi)_l^\perp/\mathrm{ad}(\xi)_l)_x \oplus (\mathrm{ad}(\xi)_l)_x$ as Lie algebras. Due to Lemma 3.2.1, $(\mathrm{ad}(\xi)_l^\perp/\mathrm{ad}(\xi)_l)_x$ is a reductive Lie algebra. Due to *(i)*, $(\mathrm{ad}(\xi)_l)_x$ is a nilpotent Lie algebra, and thus it is isomorphic to the nilpotent radical of $(\mathrm{ad}(\xi)_l^\perp)_x$.

For all $x \in X$, we have found the Levi decomposition of $(\mathrm{ad}(\xi)_l^\perp)_x$, implying that it is a parabolic Lie subalgebra of $\mathrm{ad}(\xi)_x$. Thus, the claim of *(ii)* follows. ☐

The fibers of $\mathrm{ad}(\xi)$ are isomorphic to $\mathfrak{g}$, and the fibers of $\mathrm{ad}(\xi)_l^\perp$ are isomorphic to a parabolic Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, inducing a parabolic subgroup P of G. Since the adjoint representation $\mathrm{Ad} : G \to \mathbf{GL}(\mathfrak{g})$ restricts to $\mathrm{Ad} : P \to \mathbf{GL}(\mathfrak{p})$, $\mathrm{ad}(\xi)_l^\perp$ induces a reduction $\sigma^*\xi$ of ξ to P, such that $\mathrm{ad}(\sigma^*\xi)$ is isomorphic to $\mathrm{ad}(\xi)_l^\perp$.

DEFINITION 3.2.5. A reduction $\sigma^*\xi$ of ξ to a parabolic subgroup P of G is called a *canonical reduction* if $\mathrm{ad}(\sigma^*\xi)$ is isomorphic to $\mathrm{ad}(\xi)_l^\perp$.

Knowing that canonical reductions exist, we ask to what extent they are unique, for which the following lemma is useful.

LEMMA 3.2.6. *Let P and P' be parabolic subgroups of G, let $\sigma^*\xi$ be a reduction of ξ to P, and let $\sigma'^*\xi$ be a reduction of ξ to P'. If there exists an isomorphism $\varphi : ad(\sigma^*\xi) \to ad(\sigma'^*\xi)$, then P and P' are conjugate, i.e., there exists $g \in G$, such that $conj_g : G \to G$, $h \mapsto ghg^{-1}$ fulfills $conj_g(P) = P'$.*

Through $conj_g$, we induce an extension $(\sigma^*\xi)(P')$ of $\sigma^*\xi$ to P', such that there exists an isomorphism $\psi : (\sigma^*\xi)(P') \to (\sigma'^*\xi)$ of principal-P'-bundles.

PROOF. Given $\varphi : \mathrm{ad}(\sigma^*\xi) \to \mathrm{ad}(\sigma'^*\xi)$, the fibers of the adjoint bundles are isomorphic, implying that the Lie algebras $\mathfrak{p}$ and $\mathfrak{p}'$ are isomorphic. As parabolic subgroups of G, P and P' are thus conjugate to each other.

Let $g \in G$ such that $\mathrm{conj}_g(P) = P'$. We fix an arbitrary $Y \in \mathfrak{p}$, such that $Y \neq 0$. For all $x \in X$, φ restricts to an isomorphism on the fibers at x:

$$\mathrm{ad}(\sigma^*\xi)_x = P\backslash(\sigma(x) \times \mathfrak{p}) \to \mathrm{ad}(\sigma'^*\xi)_x = P'\backslash(\sigma'(x) \times \mathfrak{p}'), \quad [v, Y] \mapsto [\eta_x(v), \mathrm{Ad}(g)(Y)], \tag{3.2.5}$$

through which we induce a map $\eta_x : \sigma(x) \to \sigma'(x)$ that induces an isomorphism $\eta : \sigma^*\xi \to \sigma'^*\xi$ of fiber bundles.

To ensure that η induces an isomorphism $\psi : (\sigma^*\xi)(P') \to (\sigma'^*\xi)$ of principal bundles, we need to verify that for all $x \in X$, η_x is P'-equivariant with respect to the conjugation conj_g. For all $v \in \sigma(x)$ and all $p \in P$, we have:

$$[\psi_x(v)gpg^{-1}, \mathrm{Ad}(g)(Y)] = [\psi_x(v), \mathrm{Ad}(gp^{-1}g^{-1}g)(Y)], \tag{3.2.6}$$
$$= [\psi_x(v), \mathrm{Ad}(gp^{-1})(Y)], \tag{3.2.7}$$
$$= [\psi_x(v)pg^{-1}, Y], \tag{3.2.8}$$
$$= [\psi_x(v)p, \mathrm{Ad}(g)(Y)]. \tag{3.2.9}$$

Thus, the claim follows. □

Due to Lemma 3.2.6, two canonical reductions $\sigma^*\xi$ and $\sigma'^*\xi$ of ξ to P and P' are conjugate in the sense that there exists an isomorphism $\psi : (\sigma^*\xi)(P') \to (\sigma'^*\xi)$ of principal-P'-bundles.

We now see some examples of canonical reductions, and how they are related to Harder-Narasimhan filtrations.

EXAMPLE 3.2.7. Let ξ be a princpal-$\mathbf{GL}(r, \mathbb{C})$-bundle, and let E_ξ be the induced vector bundle as constructed in (c) of Example 2.1.12. The Harder-Narasimhan filtration $\mathcal{F}_{E_\xi}$ of E_ξ induces a reduction $\sigma^*\xi$ of ξ to a standard parabolic subgroup P_I of $\mathbf{GL}(r, \mathbb{C})$, through Lemma 2.1.15.

We claim that $\sigma^*\xi$ is a canonical reduction. As vector bundles, $\mathrm{ad}(\xi)$ is isomorphic to the vector bundle of endomorphisms $\mathrm{End}(E_\xi)$. By restricting this isomorphism, the parabolic Lie algebra subbundle $\mathrm{ad}(\xi)_l^\perp$ from Lemma 3.2.4 is isomorphic to the subbundle of $\mathrm{End}(E_\xi)$ of endomorphisms preserving the Harder-Narasimhan filtration $\mathcal{F}_{E_\xi}$ of E_ξ. From this, the claim follows.

Similar arguments also work for special orthogonal and symplectic Harder-Narasimhan filtrations.

EXAMPLE 3.2.8. (a) Let ξ be a princpal-$\mathbf{SO}(r, \mathbb{C})$-bundle, and let (E_ξ, β, τ) be the induced special orthogonal vector bundle. The special orthogonal Harder-Narasimhan filtration $\mathcal{F}_{(E_\xi, \beta)}$ of E_ξ induces a reduction $\sigma^*\xi$ of ξ to a standard parabolic subgroup P_I of $\mathbf{SO}(r, \mathbb{C})$, through Lemma 2.1.23. This is a canonical reduction of ξ.

(b) Let ξ be a princpal-$\mathbf{Sp}(2n, \mathbb{C})$-bundle, and let (E_ξ, β) be the induced symplectic vector bundle. The symplectic Harder-Narasimhan filtration $\mathcal{F}_{(E_\xi,\beta)}$ of E_ξ induces a reduction $\sigma^*\xi$ of ξ to a standard parabolic subgroup P_I of $\mathbf{Sp}(2n, \mathbb{C})$, through Lemma 2.1.26. This is a canonical reduction of ξ.

3.2.2. The Biswas-Holla approach

Let ξ be a principal-G-bundle. The approach in [BH04] also finds conditions for reductions $\sigma^*\xi$ of ξ, to parabolic subgroups P of G, to be canonical reductions. These conditions are analogous to those for Harder-Narasimhan filtrations in Theorem 3.1.1. One condition deals with Ramanathan-semistability, and the other imposes slope-inequalities.

By investigating the reductions from Example 3.2.7 and Example 3.2.8, we can infer these conditions for canonical reductions.

LEMMA 3.2.9. *Let ξ be a princpal-$\mathbf{GL}(r, \mathbb{C})$-bundle, with the canonical reduction $\sigma^*\xi$ of ξ to P_I, from Example 3.2.7.*

(i) The extension $(\sigma^*\xi)(L_I)$ of $\sigma^*\xi$ to the Levi-factor L_I of P_I is Ramanathan-semistable.

(ii) For the length l of the Harder-Narasimhan filtration $\mathcal{F}_{E_\xi}$, and for all $m = 1, \ldots, l-1$, let $\chi_m : P_I \to \mathbb{C}^\times$ evaluate the determinant of the m-th diagonal block divided by the determinant of the $m+1$-st diagonal block. The induced line bundle $\chi_m(\sigma^*\xi)$ fulfills $deg(\chi_m(\sigma^*\xi)) > 0$.

PROOF. We prove *(i)*. As we saw in Example 1.2.2, the Levi-factor L_I consists of the diagonal block matrices of P_I, the sizes of which correspond to the ranks r_m of the quotients $(F_\xi)_m$ of the Harder-Narasimhan filtration of E_ξ. Since $L_I \simeq \mathbf{GL}(r_1, \mathbb{C}) \times \ldots \times \mathbf{GL}(r_l, \mathbb{C})$, we have due to Lemma 2.2.23 that:

$$(\sigma^*\xi)(L_I) \simeq (\sigma^*\xi)(\mathbf{GL}(r_1, \mathbb{C})) \times \ldots \times (\sigma^*\xi)(\mathbf{GL}(r_l, \mathbb{C})), \tag{3.2.10}$$

is Ramanathan-semistable if and only if for all $m = 1, \ldots, l$, $(\sigma^*\xi)(\mathbf{GL}(r_m, \mathbb{C}))$ is Ramanathan-semistable. Due to Theorem 2.2.14, this is equivalent to the induced vector bundle of $(\sigma^*\xi)(\mathbf{GL}(r_m, \mathbb{C}))$ being slope-semistable for all $m = 1, \ldots, l$, which is true since the induced vector bundles are $(F_\xi)_m$ for all $m = 1, \ldots, l$.

Now we prove *(ii)*. For all $m = 1, \ldots, l-1$, we have:

$$\begin{aligned} \deg(\chi_m(\sigma^*\xi)) &= \deg((F_\xi)_m \otimes (F_\xi)^*_{m+1}) && (3.2.11)\\ &= \deg((F_\xi)_m)r_{m+1} - \deg((F_\xi)_{m+1})r_m, && (3.2.12)\\ &> 0, && (3.2.13) \end{aligned}$$

due to the slope inequalities of the Harder-Narasimhan filtration of E_ξ. □

We now handle canonical reductions of principal-$\mathbf{SO}(r, \mathbb{C})$-bundles, using the notation of Subsection 1.2.2. Similarly to Lemma 3.2.9, this reduction fulfills a Ramanathan-semistability condition and a slope inequality condition.

LEMMA 3.2.10. *Let ξ be a princpal-$\mathbf{SO}(r, \mathbb{C})$-bundle, with the canonical reduction $\sigma^*\xi$ of ξ to P_I, from Example 3.2.8.*

(i) The extension $(\sigma^*\xi)(L_I)$ of $\sigma^*\xi$ to the Levi-factor L_I of P_I is Ramanathan-semistable.

(ii) For the length l of the Harder-Narasimhan filtration $\mathcal{F}_{(E_\xi,\beta)}$, and for all $m = 1, \ldots, l-1$, let $\overline{\chi}_m : \overline{P}_I \to \mathbb{C}^\times$ evaluate the determinant of the m-th diagonal block divided by the determinant of the $m+1$-st diagonal block. Through the

isomorphism $P_I \simeq \overline{P}_I$, this induces $\chi_m : P_I \to \mathbb{C}^\times$. The induced line bundle $\chi_m(\sigma^*\xi)$ fulfills deg$(\chi_m(\sigma^*\xi)) > 0$.

Let $\overline{\eta} : \overline{P}_I \to \mathbb{C}^\times$ evaluate the determinant of the last diagonal block, this induces $\eta : P_I \to \mathbb{C}^\times$. The induced line bundle $\eta(\sigma^*\xi)$ fulfills deg$(\eta(\sigma^*\xi)) > 0$.

PROOF. Analogous to that of Lemma 3.2.9. □

Finally, we view the case of principal-$\mathbf{Sp}(2n, \mathbb{C})$-bundles, using the notation of Subsection 1.2.3.

LEMMA 3.2.11. *Let ξ be a princpal-$\mathbf{Sp}(2n, \mathbb{C})$-bundle, with the canonical reduction $\sigma^*\xi$ of ξ to P_I, from Example 3.2.8.*

(i) The extension $(\sigma^*\xi)(L_I)$ of $\sigma^*\xi$ to the Levi-factor L_I of P_I is Ramanathan-semistable.

(ii) For the length l of the Harder-Narasimhan filtration $\mathcal{F}_{(E_\xi,\beta)}$, and for all $m = 1, \dots, l-1$, let $\overline{\chi}_m : \overline{P}_I \to \mathbb{C}^\times$ evaluate the determinant of the m-th diagonal block divided by the determinant of the $m+1$-st diagonal block. Through the isomorphism $P_I \simeq \overline{P}_I$, this induces $\chi_m : P_I \to \mathbb{C}^\times$. The induced line bundle $\chi_m(\sigma^*\xi)$ fulfills deg$(\chi_m(\sigma^*\xi)) > 0$.

Let $\overline{\eta} : \overline{P}_I \to \mathbb{C}^\times$ evaluate the determinant of the last diagonal block, this induces $\eta : P_I \to \mathbb{C}^\times$. The induced line bundle $\eta(\sigma^*\xi)$ fulfills deg$(\eta(\sigma^*\xi)) > 0$.

PROOF. Analogous to that of Lemma 3.2.9 and Lemma 3.2.10. □

In order to use these lemmas to characterize canonical reductions, we first need to understand dominant characters, for which we review some important results on characters of complex Lie groups.

DEFINITION 3.2.12. For a complex Lie group H, a morphism $\chi : H \to \mathbb{C}^\times$ of complex Lie groups is called a *character* of H. We denote the *character group* of H by $\mathbf{X}(H) = \mathrm{Hom}(H, \mathbb{C}^\times)$, which has a natural group structure induced from $\mathbb{C}^\times$.

REMARK 3.2.13. Let $\chi : H \to \mathbb{C}^\times$ be a character.

(a) Since $\mathbb{C}^\times$ is abelian, a character χ factorizes through the *abelianization* $H_{ab} = H/H_{der}$ of H. Thus, we have an isomorphism $\mathbf{X}(H) \simeq \mathbf{X}(H_{ab})$ of groups.

(b) We have the commutative diagram:

$$\begin{array}{ccc} H & \xrightarrow{\quad\chi\quad} & \mathbb{C}^\times \\ {\scriptstyle \exp_H}\uparrow & & \uparrow{\scriptstyle \exp_{\mathbb{C}^\times}} \\ \mathfrak{h} & \xrightarrow{\quad D\chi\quad} & \mathbb{C} \end{array} \quad . \tag{3.2.14}$$

If H is connected, the map $\chi \mapsto D\chi$ is an injection from $\mathbf{X}(H)$ into $\mathfrak{h}^\vee$, as a consequence of [Čap23, Theorem 1.9].

Thus, for connected complex reductive groups G, their character groups $\mathbf{X}(G)$ embed as lattices of $\mathfrak{z}(\mathfrak{g})^\vee$, since the abelianization G_{ab} has the Lie algebra $\mathfrak{g}_{ab} = \mathfrak{g}/\mathfrak{g}_{ss} \simeq \mathfrak{z}(\mathfrak{g})$.

REMARK 3.2.14. Using (a) of Remark 2.2.20, we have the isomorphism of complex algebraic varieties:

$$G_{ab} = G/G_{der} = (G_{der}\mathbf{R}(G))/G_{der} \simeq \mathbf{R}(G)/(G_{der} \cap \mathbf{R}(G)). \tag{3.2.15}$$

Since $\mathbf{R}(G)$ is a torus, due to [MT12, Proposition 6.20], and since $G_{der} \cap \mathbf{R}(G)$ is finite, $G_{ab} \simeq (\mathbb{C}^\times)^m$ is also a torus. We thus have the isomorphisms of groups:

$$\mathbf{X}(G) \simeq \mathbf{X}(G_{ab}) \simeq \mathbf{X}((\mathbb{C}^\times)^m) \simeq \mathbb{Z}^m. \tag{3.2.16}$$

In order to define dominant characters, we look at characters of standard parabolic subgroups P_I of G, with respect to a fixed Cartan subgroup T of G, and a Borel subgroup B of G, containing T.

REMARK 3.2.15. A character $\chi : P_I \to \mathbb{C}^\times$ of a standard parabolic subgroup P_I of G forms an irreducible one-dimensional representation.

In the Levi decomposition $P_I \simeq U_I \rtimes L_I$, the restriction $\chi|_{U_I} : U_I \to \mathbb{C}^\times \simeq \mathbf{GL}(\mathbb{C})$ maps to unipotent elements of $\mathbf{GL}(\mathbb{C})$, which are trivial. Thus, χ factorizes through the Levi-factor L_I. Using Remark 3.2.13, the restriction $\chi|_{L_I} : L_I \to \mathbb{C}^\times$ corresponds to $D\chi|_{\mathfrak{z}(\mathfrak{l}_I)} : \mathfrak{z}(\mathfrak{l}_I) \to \mathbb{C}$ in $\mathfrak{z}(\mathfrak{l}_I)^\vee$. In particular, $\chi : P_I \to \mathbb{C}^\times$ can be determined by its restriction $D\chi|_{\mathfrak{t}} : \mathfrak{t} \to \mathbb{C}$ in $\mathfrak{t}^\vee$.

DEFINITION 3.2.16. Let P_I be a standard parabolic subgroup of G. A nontrivial character $\chi : P_I \to \mathbb{C}^\times$ is a *dominant character* if $D\chi|_{\mathfrak{t}} : \mathfrak{t} \to \mathbb{C}$ is a nonnegative integer linear combination of simple roots in $\triangle$.

The characters we constructed in *(ii)* of Lemma 3.2.9, Lemma 3.2.10 and Lemma 3.2.11 are dominant characters, with respect to our standard choices of Cartan and Borel subgroups of $\mathbf{GL}(r,\mathbb{C})$, $\mathbf{SO}(r,\mathbb{C})$ and $\mathbf{Sp}(2n,\mathbb{C})$ from Chapter 1. We verify an explicit example.

EXAMPLE 3.2.17. Let P_I the standard parabolic subgroup of $\mathbf{GL}(4,\mathbb{C})$ corresponding to $I = \{\alpha_{2,3}\} \subset \triangle$, using the notation of Subsection 1.2.1. The character:

$$\chi_2 : P_I \to \mathbb{C}^\times, \quad \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mapsto \det(A)/\det(D), \tag{3.2.17}$$

is a dominant character, since $D\chi_2|_{\mathfrak{t}} = \alpha_{1,2} + 2\alpha_{2,3} + \alpha_{3,4}$.

We can now formulate an alternate characterization of canonical reductions of principal-G-bundles ξ, following [BH04]. Let T be a Cartan subgroup of G, and let B be a Borel subgroup of G, containing T.

THEOREM 3.2.18. *A reduction $\sigma^*\xi$ of ξ to a standard parabolic subgroup P_I of G is a canonical reduction if and only if:*

(i) The extension $(\sigma^*\xi)(L_I)$ of $\sigma^*\xi$ to the Levi-factor L_I is Ramanathan-semistable.
(ii) For all dominant characters $\chi : P_I \to \mathbb{C}^\times$, we have deg$(\chi(\sigma^*\xi)) > 0$.

This theorem will be proven at the end of this subsection.

The first property *(i)* generalizes the Ramanathan-semistability of $(\sigma^*\xi)(L_I)$ from Lemma 3.2.9, Lemma 3.2.10 and Lemma 3.2.11. The second property *(ii)* also generalizes the slope inequality conditions from these lemmas, however, in explicit cases, condition *(ii)* may be difficult to verify for all dominant characters. Luckily, the following lemma explains that we only need to check this condition for finitely many characters.

LEMMA 3.2.19. *Using the notation of Theorem 3.2.18, we have the following:*

(i) For all $\alpha \in I$, there exists a nontrivial character $\chi_\alpha : P_I \to \mathbb{C}^\times$, such that $D\chi_\alpha|_{\mathfrak{t}} : \mathfrak{t} \to \mathbb{C}$ is an integer linear combination of elements in $\triangle$, whose coefficient in α is positive, and whose coefficients in $I \setminus \{\alpha\}$ are 0.
(ii) For a reduction $\sigma^*\xi$ of ξ to a standard parabolic subgroup P_I of G, fulfilling (i) of Theorem 3.2.18, condition (ii) of Theorem 3.2.18 is fulfilled if and only if for all $\alpha \in I$, there exists a character χ_α from (i) such that deg$(\chi_\alpha(\sigma^*\xi)) > 0$.

PROOF. See the middle section of the proof of [BH04, Proposition 3.1]. □

Example 3.2.17 gives an example a of character χ_α as in this lemma. Note that in general, χ_α is not necessarily a dominant character.

In order to prove Theorem 3.2.18, we first prove that a reduction $\sigma^*\xi$ of ξ, fulfilling *(i)* and *(ii)* of Theorem 3.2.18, exists. The following lemma helps construct a candidate for such a reduction.

LEMMA 3.2.20. *(i) Let:*

$$deg_{\max}(\xi) = \sup \left\{ deg(ad(\sigma^*\xi)) \middle| \begin{array}{l} P_I \text{ is a standard parabolic subgroup of } G, \\ \sigma^*\xi \text{ is a reduction of } \xi \text{ to } P_I \end{array} \right\}. \quad (3.2.18)$$

We have $deg_{\max}(\xi) < \infty$, and there exists reductions $\sigma^*\xi$ of ξ to P_I, such that $deg(ad(\sigma^*\xi)) = deg_{\max}(\xi)$.

(ii) Let:

$$\mathcal{P}_\xi = \left\{ P_I \text{ is a standard parabolic subgroup of } G \middle| \begin{array}{l} \exists \sigma^*\xi \text{ reduction of } \xi \text{ to } P_I, \\ deg(ad(\sigma^*\xi)) = deg_{\max}(\xi) \end{array} \right\}. \quad (3.2.19)$$

We claim that there exists a standard parabolic subgroup $P_I \in \mathcal{P}_\xi$ that is maximal in terms of inclusion.

PROOF. We prove *(i)*. Due to the short exact sequence in (2.2.7) from Remark 2.2.7, for all reductions $\sigma^*\xi$ of ξ to a standard parabolic subgroup P_I of G, $\mathrm{ad}(\sigma^*\xi)$ is isomorphic to a subbundle of $\mathrm{ad}(\xi)$. Due to Lemma 3.1.4, we know that the degrees of subbundles of $\mathrm{ad}(\xi)$ are bounded from above, i.e., $\deg_{\max}(\mathrm{ad}(\xi)) < \infty$. Thus, *(i)* follows.

Since $\mathcal{P}_\xi$ is nonempty, we can clearly find a maximal element in $\mathcal{P}_\xi$, proving *(ii)*. □

The following theorem is taken from [BH04, Proposition 3.1].

THEOREM 3.2.21. *There exists a reduction $\sigma^*\xi$ of ξ to a standard parabolic subgroup P_I of G, such that:*

(i) The degree $deg(ad(\sigma^*\xi))$ is equal to $deg_{\max}(\xi)$.

(ii) The parabolic subgroup P_I is maximal, in terms of inclusion, within $\mathcal{P}_\xi$.

Then $\sigma^*\xi$ is a reduction of ξ to P_I fulfilling (i) and (ii) of Theorem 3.2.18.

PROOF. Due to Lemma 3.2.20, a reduction $\sigma^*\xi$ of ξ to P_I exists, fulfilling *(i)* and *(ii)*.

We first show condition *(i)* of Theorem 3.2.18, i.e., that $(\sigma^*\xi)(L_I)$ is Ramanathan-semistable. Note that L_I is a complex reductive group, containing T and the Borel subgroup $L_I \cap B$. Assume to the contrary, i.e., there exists a reduction $\sigma_1'^*((\sigma^*\xi)(L_I))$ of $(\sigma^*\xi)(L_I)$ to a maximal standard parabolic subgroup P_1' of L_I, such that $\deg(\sigma_1'^* V_{(\sigma^*\xi)(L_I)/P_1'}) < 0$.

Let P_1 be the preimage of P_1' of the projection $P_I \to P_I/U_I \simeq L_I$. We have that G/P_1 is a complete variety, since the projection $G/P_1 \to G/P_I$ is a fiber bundle over the base space G/P_I, which is a complete variety, and with fibers isomorphic to $P_I/P_1 \simeq L_I/P_1'$, which are complete varieties. Thus, P_1 is a standard parabolic subgroup of G. Using $P_I/P_1 \simeq L_I/P_1'$, we find an injective morphism of fiber bundles $\varphi : (\sigma^*\xi)(L_I)/P_1' \to (\sigma^*\xi)/P_1$, given on the fibers $x \in X$ by:

$$((\sigma^*\xi)(L_I)/P_1')_x = (P_I \backslash (\sigma(x) \times L_I))/P_1' \to ((\sigma^*\xi)/P_1)_x = \sigma(x)/P_1, [v, g] \mapsto vgP_1. \quad (3.2.20)$$

Thus, the reduction $\sigma_1'^*((\sigma^*\xi)(L_I))$ of $(\sigma^*\xi)(L_I)$ induces a reduction $\sigma_1^*\xi$ of ξ to P_1.

We observe the following short exact sequence:

$$0 \to \mathfrak{p}_1 \to \mathfrak{p}_I \to \mathfrak{p}_I/\mathfrak{p}_1 \to 0, \quad (3.2.21)$$

with the adjoint representations $\mathrm{Ad} : P_I \to \mathbf{GL}(\mathfrak{p}_1)$, $\mathrm{Ad} : P_I \to \mathbf{GL}(\mathfrak{p}_I)$ and $\mathrm{Ad} : P_I \to \mathbf{GL}(\mathfrak{p}_I/\mathfrak{p}_1)$. From this, we induce a short exact sequence of vector bundles:

$$0 \to \mathrm{ad}(\sigma_1^*\xi) \to \mathrm{ad}(\sigma^*\xi) \to \sigma_1'^* V_{(\sigma^*\xi)(L_I)/P_1'} \to 0. \quad (3.2.22)$$

Due to the additivity of the degree, it follows:

$$\deg(\mathrm{ad}(\sigma^*\xi)) = \deg(\mathrm{ad}(\sigma_1^*\xi)) + \deg(\sigma_1'^* V_{(\sigma^*\xi)(L_I)/P_1'}) < \deg(\mathrm{ad}(\sigma_1^*\xi)). \tag{3.2.23}$$

Thus, $\sigma_1^*\xi$ is a reduction of ξ to P_1 that contradicts the maximality property $\deg(\mathrm{ad}(\sigma^*\xi)) = \deg_{\max}(\xi)$ from *(i)*, and the assumption is false.

We now show condition *(ii)* of Theorem 3.2.18. Using Lemma 3.2.19, for $\alpha \in I$, it suffices to show $\deg(\chi_\alpha(\sigma^*\xi)) > 0$. If $P_I = G$, then $I = \emptyset$ and there is nothing to show. Otherwise, let $P_2 = P_{I\setminus\{\alpha\}}$ be the standard parabolic subgroup of G properly containing P_I. We denote by P_2' the image of P_I of the projection $P_2 \to P_2/U_2 \simeq L_2$ to the Levi-factor L_2. Note that L_2 is a complex reductive group, containing T and the Borel subgroup $L_2 \cap B$. Since $P_2/P_I \simeq L_2/P_2'$ is a complete variety, P_2' is a standard parabolic subgroup of L_2. As P_I is contained within P_2, the section $\sigma : X \to \xi/P_I$ induces a section $\sigma_2 : X \to \xi/P_2$. Moreover, we can define a section $\sigma_2' : X \to (\sigma^*\xi)(L_2)/P_2'$ explicitly:

$$\sigma_2' : x \mapsto [\sigma(x), e]P_2'. \tag{3.2.24}$$

We observe the following short exact sequence:

$$0 \to \mathfrak{p}_I \to \mathfrak{p}_2 \to \mathfrak{p}_2/\mathfrak{p}_I \to 0, \tag{3.2.25}$$

with the adjoint representations $\mathrm{Ad} : P_2 \to \mathbf{GL}(\mathfrak{p}_I)$, $\mathrm{Ad} : P_2 \to \mathbf{GL}(\mathfrak{p}_2)$ and $\mathrm{Ad} : P_2 \to \mathbf{GL}(\mathfrak{p}_2/\mathfrak{p}_I)$. From this, we induce a short exact sequence of vector bundles:

$$0 \to \mathrm{ad}(\sigma^*\xi) \to \mathrm{ad}(\sigma_2^*\xi) \to \sigma_2'^* V_{(\sigma^*\xi)(L_2)/P_2'} \to 0, \tag{3.2.26}$$

Due to the additivity of the degree, it follows:

$$\deg(\mathrm{ad}(\sigma_2^*\xi)) = \deg(\mathrm{ad}(\sigma^*\xi)) + \deg(\sigma_2'^* V_{(\sigma^*\xi)(L_2)/P_2'}). \tag{3.2.27}$$

Since P_2 properly contains P_I, we have $P_2 \notin \mathcal{P}_\xi$ and $\deg(\sigma_2'^* V_{(\sigma^*\xi)(L_2)/P_2'}) < 0$.

We define $\chi = \det \circ \mathrm{Ad} : P_I \to \mathbb{C}^\times$, using $\mathrm{Ad} : P_I \to \mathbf{GL}(\mathfrak{p}_2/\mathfrak{p}_I)$. With respect to $\mathrm{ad} : \mathfrak{p}_I \to \mathbf{Der}(\mathfrak{p}_2/\mathfrak{p}_I)$, the weights of $\mathfrak{p}_2/\mathfrak{p}_I$ are of the form $\gamma \in \mathfrak{t}^\vee$, with a negative integer coefficient in α, and coefficients 0 in $I\setminus\{\alpha\}$. Therefore, χ is of the form $1/\chi_\alpha$ from Lemma 3.2.19. As an irreducible representation, χ factorizes through L_I, and thus we have $\det(\sigma_2'^* V_{(\sigma^*\xi)(L_2)/P_2'}) \simeq ((1/\chi_\alpha)(\sigma^*\xi))$, and the claim $\deg(\chi_\alpha(\sigma^*\xi)) > 0$ follows. □

Having constructed a reduction $\sigma^*\xi$ of ξ to P_I that fulfills conditions *(i)* and *(ii)* of Theorem 3.2.18, we wish to prove that this reduction is unique up to conjugation, in the sense of Lemma 3.2.6. We can then finally prove that conditions *(i)* and *(ii)* of Theorem 3.2.18 are equivalent to canonical reductions, i.e., that Theorem 3.2.18 holds.

REMARK 3.2.22. Let $\rho : G \to \mathbf{GL}(V)$ be a finite-dimensional irreducible representation, such that the center $\mathbf{Z}(G)^0$ maps to scalar multiples of the identity in $\mathbf{GL}(V)$. By viewing $\mathbf{GL}(V)$ as a complex reductive group isomorphic to $\mathbf{GL}(r, \mathbb{C})$, it is proven in [RR83, Theorem 3.18] that if ξ is Ramanathan-semistable, then $\xi(V)$ is slope-semistable.

The following theorem is taken from [BH04, Theorem 4.1].

THEOREM 3.2.23. *For reductions $\sigma^*\xi$ of ξ to P_I and $\sigma'^*\xi$ of ξ to $P_{I'}$, fulfilling (i) and (ii) of Theorem 3.2.18, there exists a conjugation conj_g, for $g \in G$, such that $\mathrm{conj}_g(P) = P'$, and an isomorphism $\psi : (\sigma^*\xi)(P') \to \sigma'^*\xi$.*

PROOF. Due to Lemma 3.2.6, it suffices to show that $\mathrm{ad}(\sigma^*\xi) \simeq \mathrm{ad}(\sigma'^*\xi)$. Let $E = \mathrm{ad}(\xi)$, $E'' = \mathrm{ad}(\sigma^*\xi)$ and $E' = \sigma'^* V_{\xi/P_{I'}}$, then due to the short exact sequence in (2.2.7) from Remark 2.2.7, E'' is isomorphic to a subbundle of E, and E' is isomorphic to a quotient of E. We write a sequence with inclusion and quotient morphisms:

$$0 \to E'' \to E \to E' \to 0, \tag{3.2.28}$$

and claim that the composed map $E'' \to E'$ is 0.

We can find a filtration of subspaces of $\mathfrak{p}_I$:

$$0 = \mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_l = \mathfrak{p}_I, \tag{3.2.29}$$

such that for $m = 1, \ldots, l$, the representations $\mathrm{Ad} : P_I \to \mathbf{GL}(\mathfrak{p}_m/\mathfrak{p}_{m-1})$ are well-defined and irreducible. Since for $m = 1, \ldots, l$, $\mathrm{Ad} : P_I \to \mathbf{GL}(\mathfrak{p}_m/\mathfrak{p}_{m-1})$ irreducible, it factorizes through the Levi-factor L_I of P_I. We can use the adjoint representations $\mathrm{Ad} : P_I \to \mathbf{GL}(\mathfrak{p}_m)$, $m = 0, \ldots, l$, on (3.2.29), to induce a filtration by subbundles of E'':

$$0 = E''_0 \subsetneq \ldots \subsetneq E''_l = E'', \tag{3.2.30}$$

with the quotients $F''_m = E''_m/E''_{m-1}$, $m = 1, \ldots, l$. Due to Remark 3.2.22, the Ramanathan-semistability of $(\sigma^*\xi)(L_I)$ implies that for $m = 1, \ldots, l$, F''_m is slope-semistable.

We apply a similar argument to E'. We can find a filtration of subspaces of $\mathfrak{g}/\mathfrak{p}_{I'}$:

$$0 = \mathfrak{g}_0/\mathfrak{p}_{I'} \subsetneq \ldots \subsetneq \mathfrak{g}_{l'}/\mathfrak{p}_{I'} = \mathfrak{g}/\mathfrak{p}_{I'}, \tag{3.2.31}$$

such that for all $m = 1, \ldots, l'$, the representations $\mathrm{Ad} : P_{I'} \to \mathbf{GL}(\mathfrak{g}_m/\mathfrak{g}_{m-1})$ are well-defined and irreducible. These adjoint representations factorize through the Levi-factor $L_{I'}$ of $P_{I'}$. We can use the adjoint representations $\mathrm{Ad} : P_{I'} \to \mathbf{GL}(\mathfrak{g}_m/\mathfrak{p}_{I'})$, $m = 0, \ldots, l'$, on (3.2.31), to induce a filtration by subbundles of E':

$$0 = E'_0 \subsetneq \ldots \subsetneq E'_{l'} = E', \tag{3.2.32}$$

with the quotients $F'_m = E'_m/E'_{m-1}$, $m = 1, \ldots, l'$. Due to Remark 3.2.22, the Ramanathan-semistability of $(\sigma^*\xi)(L_I)$ implies that for $m = 1, \ldots, l'$, F'_m is slope-semistable.

The claim that $E'' \to E'$ from (3.2.28) is 0, is proven, if we show that $\mu(F''_m) \geq 0$, for $m = 1, \ldots, l$, and $\mu(F'_m) < 0$, for $m = 1, \ldots, l'$, and then apply Lemma 3.2.3.

For $m = 1, \ldots, l$, it suffices to show that $\deg(F''_m) \geq 0$. We define $\chi'' = \det \circ \mathrm{Ad} : P_I \to \mathbb{C}^\times$, for $\mathrm{Ad} : P_I \to \mathbf{GL}(\mathfrak{p}_m/\mathfrak{p}_{m-1})$. Since the weight spaces of $\mathfrak{p}_m/\mathfrak{p}_{m-1}$ are of the form $\mathfrak{g}_\gamma$, where γ is a dominant character, we can follow that:

$$\deg(F''_m) = \deg(\chi''(\sigma^*\xi)) > 0. \tag{3.2.33}$$

For $m = 1, \ldots, l'$, the argument for $\mu(F'_m) < 0$ is analogous. We define $\chi' = \det \circ \mathrm{Ad} : P_{I'} \to \mathbb{C}^\times$, for $\mathrm{Ad} : P_{I'} \to \mathbf{GL}(\mathfrak{g}_m/\mathfrak{g}_{m-1})$. Since the weight spaces of $\mathfrak{g}_m/\mathfrak{g}_{m-1}$ are of the form $\mathfrak{g}_\gamma$, where $1/\gamma$ is a dominant character, we can follow that:

$$\deg(F'_m) = \deg(\chi'(\sigma'^*\xi)) < 0. \tag{3.2.34}$$

We have proven the claim that $E'' \to E'$, from (3.2.28), is 0. Using the short exact sequence from (2.2.7), this ensures that $\mathrm{ad}(\sigma^*\xi) = E''$ is isomorphic to a subbundle of $\mathrm{ad}(\sigma'^*\xi)$. By swapping the roles of the two canonical reductions, we follow $\mathrm{ad}(\sigma^*\xi) \simeq \mathrm{ad}(\sigma'^*\xi)$, from which we find an isomorphism $\psi : (\sigma^*\xi)(P') \to \sigma'^*\xi$. □

Finally, we can prove Theorem 3.2.18, and show that the Atiyah-Bott and Biswas-Holla approaches lead to the same notion of canonical reductions.

PROOF OF THEOREM 3.2.18. It suffices to show that a canonical reduction fulfills conditions *(i)* and *(ii)* of Theorem 3.2.18. The other implication follows from the existence and uniqueness of canonical reductions, and the existence and uniqueness of reductions fulfilling *(i)* and *(ii)* of Theorem 3.2.18.

Due to Theorem 3.2.21, it also suffices to show that for a canonical reduction $\sigma^*\xi$ of ξ to P_I, we have $\deg(\mathrm{ad}(\sigma^*\xi)) = \deg_{\max}(\xi)$, and that the parabolic subgroup P_I is maximal, in terms of inclusion, within $\mathcal{P}_\xi$. Both claims follow from the fact that $\mathrm{ad}(\sigma^*\xi)$ is isomorphic to $\mathrm{ad}(\xi)_l^\perp$ in the filtration from (3.2.2). □

CHAPTER 4

Harder-Narasimhan types

In this final chapter, we learn how canonical reductions can be classified by obstruction classes and Harder-Narasimhan types. We will then calculate Harder-Narasimhan types for the $\mathbf{GL}(r,\mathbb{C})$, $\mathbf{SO}(r,\mathbb{C})$ and $\mathbf{Sp}(2n,\mathbb{C})$ cases.

To do this, we first review real forms K of connected complex reductive groups G, which are maximal compact real Lie subgroups of G. Then, we find algebraic characterizations of the fundamental groups of T, G, G_{der} and G_{ab} as $\mathbb{Z}$-modules. Using Čech cohomology, we are then able to define obstruction classes and topological types of principal bundles.

Let X be a compact connected Riemann surface and let ξ be a principal-G-bundle.

4.1. Fundamental groups of reductive groups

We follow [Hum72, 25] and [Kna88, IV.4, VI.1] to investigate real forms of G, and its Lie algebra $\mathfrak{g}$.

Definition 4.1.1. (a) Let $\mathfrak{k}$ be a real Lie subalgebra of $\mathfrak{g}$, whose complexification $\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathfrak{g}$, and whose Killing-form $\kappa_{\mathfrak{k}}$, a restriction of the Killing-form κ of $\mathfrak{g}$, is negative-semidefinite. We call $\mathfrak{k}$ a *compact real form* of $\mathfrak{g}$.

(b) Let K be a maximal compact real Lie subgroup of G, then K is a *real form* of G.

Remark 4.1.2. By constructing Chevalley bases, as seen in [Hum72, 25], compact real forms $\mathfrak{k}$ of $\mathfrak{g}$ exist, and are fully determined by a choice of Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$.

For a compact real form $\mathfrak{k}$ of $\mathfrak{g}$, we denote K as the group generated by the exponential map $\exp_G : \mathfrak{g} \to G$, which we claim is a real form. Using [Kna88, IV.4 Proposition 4.23-4.27], we follow that the group $\mathrm{Ad}(K)$, induced by $\mathrm{Ad} : K \to \mathbf{GL}(\mathfrak{k})$, is a compact real Lie subgroup of $\mathbf{GL}(\mathfrak{g})$. Thus, K is a compact real subgroup of G. To show that K is a maximal compact real Lie subgroup of G, we use that $\mathfrak{k}$ is maximal amongst real Lie subalgebras of $\mathfrak{g}$, whose Killing-form $\kappa_{\mathfrak{k}}$ is negative-semidefinite.

Example 4.1.3. The real Lie group of unitary matrices $\mathbf{U}(r)$ has the Lie algebra $\mathfrak{u}(r)$ of skew-Hermitian matrices, which has a negative-semidefinite Killing-form. Since the complexification of $\mathfrak{u}(r)$ is $\mathfrak{gl}(r,\mathbb{C})$. We follow that $\mathbf{U}(r)$ is a real form of $\mathbf{GL}(r,\mathbb{C})$. A similar argument shows that $\mathbf{SU}(r)$ is a real form of $\mathbf{SL}(r,\mathbb{C})$.

The above results were stated abstractly, however, real forms can be constructed in a concrete way for most complex reductive groups of interest, by using polar decomposition.

Remark 4.1.4. For a matrix group G closed under conjugate-transposition, there exists a complex vector space of skew-Hermitian matrices $\mathfrak{u}$, such that $\mathfrak{g}$ is the complexification of $\mathfrak{u}$ as a complex vector space. We can decompose $\mathfrak{u}$ as $\mathfrak{u} = \mathfrak{a} + \mathbf{i}\mathfrak{b}$, where $\mathfrak{a}$ is a set of skew-symmetric matrices, and $\mathfrak{b}$ is a set of symmetric matrices. We then get $\mathfrak{k} = \mathfrak{a} + \mathfrak{b}$, and $\mathfrak{g} = \mathfrak{k} + \mathbf{i}\mathfrak{k}$ is a complexification of $\mathfrak{k}$ as a Lie algebra.

We fix a Cartan subgroup $T \simeq (\mathbb{C}^{\times})^r$ of G, and the induced real form K of G. For $H = K \cap T$, we have that $H \simeq \mathbf{U}(1)^r$ is a maximal compact torus of K.

As mentioned before, we want algebraic characterizations of the fundamental groups of T, G, G_{der}, and G_{ab}, as $\mathbb{Z}$-modules. Some of these fundamental groups embed naturally as lattices of the real vector spaces:

$$\mathfrak{g}_{\mathbb{R}} = \mathbf{i}\mathfrak{k}, \qquad \mathfrak{t}_{\mathbb{R}} = \mathfrak{t} \cap \mathfrak{g}_{\mathbb{R}}, \qquad \mathfrak{z}(\mathfrak{g})_{\mathbb{R}} = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\mathbb{R}}, \tag{4.1.1}$$

where $\mathfrak{h} = \mathfrak{k} \cap \mathfrak{t} = \mathbf{i}\mathfrak{g}_{\mathbb{R}} \cap \mathfrak{t} = \mathbf{i}\mathfrak{t}_{\mathbb{R}}$.

REMARK 4.1.5. Due to $T \simeq (\mathbb{C}^{\times})^r$ and $H \simeq \mathbf{U}(1)^r$, we have isomorphisms:

$$\pi_1(T) \simeq \pi_1(H) \simeq \mathrm{Hom}(\mathbf{U}(1), H) \simeq \mathbb{Z}^r, \tag{4.1.2}$$

where $\mathrm{Hom}(\mathbf{U}(1), H)$ are the *cocharacters* of H. Since $\mathbf{U}(1)$ and H are connected, we have an injection from $\mathrm{Hom}(\mathbf{U}(1), H)$ to the derivatives:

$$\mathrm{Hom}(\mathfrak{u}(1), \mathfrak{h}) = \mathrm{Hom}_{\mathbb{R}}(\mathbf{i}\mathbb{R}, \mathbf{i}\mathfrak{t}_{\mathbb{R}}) \simeq \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \mathfrak{t}_{\mathbb{R}}) \simeq \mathfrak{t}_{\mathbb{R}}, \tag{4.1.3}$$

where the last isomorphism is given by $[2\pi \mapsto X] \mapsto X$. Thus, $\pi_1(T)$ is isomorphic to the full-rank lattice $\Gamma = \{X \in \mathfrak{t}_{\mathbb{R}} | \exp_H(2\pi\mathbf{i}X) = e\}$ of $\mathfrak{t}_{\mathbb{R}} \simeq \mathbb{R}^r$, called the *kernel lattice*.

We now recall some constructions from abstract root systems, applied to $\Phi(\mathfrak{g}, \mathfrak{t})$, which is a root system of $(V_{\mathbb{R}}, \langle \_, \_ \rangle)$ from Theorem 1.1.13.

DEFINITION 4.1.6. For a root $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$:

(a) The *coroot* of α is $\alpha^* = 2\alpha/\langle \alpha, \alpha \rangle \in \mathfrak{t}^{\vee}$. The set of coroots is denoted by $\Phi^*(\mathfrak{g}, \mathfrak{t})$. In [Hum72, III 9.2], it is shown that $\Phi^*(\mathfrak{g}, \mathfrak{t})$ is a root system of $(V_{\mathbb{R}}, \langle \_, \_ \rangle)$.

(b) The *dual-root* of α is $H_{\alpha|_{\mathfrak{g}_{ss} \cap \mathfrak{t}}} \in \mathfrak{g}_{ss} \cap \mathfrak{t}$, using notation from (1.1.17). The set of dual-roots is denoted by $\Phi_H(\mathfrak{g}, \mathfrak{t})$. Let $W_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}(\Phi_H(\mathfrak{g}, \mathfrak{t}))$, by construction, $\Phi_H(\mathfrak{g}, \mathfrak{t})$ is a root system of $(W_{\mathbb{R}}, \kappa|_{W_{\mathbb{R}} \times W_{\mathbb{R}}})$, of the same type as $\Phi(\mathfrak{g}, \mathfrak{t})$.

(c) The *dual-coroot* of α is $H_{\alpha^*|_{\mathfrak{g}_{ss} \cap \mathfrak{t}}} = H^*_{\alpha|_{\mathfrak{g}_{ss} \cap \mathfrak{t}}} \in \mathfrak{g}_{ss} \cap \mathfrak{t}$. The set $\Phi^*_H(\mathfrak{g}, \mathfrak{t})$ of dual-coroots forms a root system of $(W_{\mathbb{R}}, \kappa|_{W_{\mathbb{R}} \times W_{\mathbb{R}}})$, of the same type as $\Phi^*(\mathfrak{g}, \mathfrak{t})$.

REMARK 4.1.7. (a) The *dual-coroot lattice* $\Lambda = \mathrm{span}_{\mathbb{Z}}(\Phi^*_H(\mathfrak{g}, \mathfrak{t}))$ is contained within Γ, as proven in [Hal15, Lemma 12.8].

(b) Using exponential maps on $\mathbf{R}(G)$ and G_{ab}, $T \simeq (\mathbb{C}^{\times})^r$ induces fixed isomorphisms $G_{ab} \simeq \mathbf{R}(G) \simeq (\mathbb{C}^{\times})^m$, where $0 \leq m \leq r$.

We can now search for expressions of $\pi_1(G)$, $\pi_1(G_{der})$ and $\pi_1(G_{ab})$.

LEMMA 4.1.8. *(i) The inclusion $T \hookrightarrow G$ induces a surjection $\pi_1(T) \to \pi_1(G)$, such that $\pi_1(G) \simeq \Gamma/\Lambda$.*

Let $\widehat{\Lambda}$ be the saturation of Λ in Γ, i.e., the sublattice $\widehat{\Lambda}$ of Γ containing Λ, minimal with respect to inclusion, such that $\Gamma/\widehat{\Lambda}$ is free. We claim that:

(ii) The group $\pi_1(G_{ab}) \simeq \mathbb{Z}^m$ is a lattice, and the group $\pi_1(G_{der})$ is finite.

(iii) The short exact sequence $1 \to G_{der} \to G \to G_{ab} \to 1$ induces the short exact sequence:

$$1 \to \pi_1(G_{der}) \to \pi_1(G) \to \pi_1(G_{ab}) \to 1, \tag{4.1.4}$$

of fundamental groups. In particular, $\pi_1(G_{ab}) \simeq \Gamma/\widehat{\Lambda}$ and $\pi_1(G_{der}) \simeq \widehat{\Lambda}/\Lambda$.

PROOF. For the proof of *(i)*, see [Hal15, Corollary 13.18], noting that $T \hookrightarrow G$ is the complexified version of $H \hookrightarrow K$. In particular, the surjectivity of $\pi_1(T) \to \pi_1(G)$ is proven in [Hal15, Proposition 13.37].

We prove *(ii)*. Since $G_{ab} \simeq (\mathbb{C}^{\times})^m$, we have that $\pi_1(G_{ab}) \simeq \mathbb{Z}^m$ is a lattice. As G_{der} has a real form K_{der} that is semisimple, Lie algebra cohomology implies that $\pi_1(G_{der})$ is finite, as seen in [CE48, Theorem 16.1].

We now prove *(iii)*. Since $G \to G_{ab}$ is a fiber bundle, with fibers isomorphic to G_{der}, it is a fibration, inducing the long exact homotopy sequence:

$$\ldots \to \pi_2(G_{ab}) \to \pi_1(G_{der}) \to \pi_1(G) \to \pi_1(G_{ab}) \to \pi_0(G_{der}) \to \ldots \tag{4.1.5}$$

Since G_{der} is connected, we have $\pi_0(G_{der}) \simeq 1$. Using Morse theory, it can be proven that $\pi_2(G_{ab}) \simeq 1$, as G_{ab} is a Lie group. Therefore, the sequence in (4.1.4) is exact. The isomorphisms $\pi_1(G_{ab}) \simeq \Gamma/\widehat{\Lambda}$, and $\pi_1(G_{der}) \simeq \widehat{\Lambda}/\Lambda$, follow directly from (4.1.4) and *(ii)*. □

REMARK 4.1.9. (a) For a real form K_{ab} of G_{ab}, we have:

$$\mathbf{X}(G) \simeq \mathbf{X}(G_{ab}) \simeq \mathrm{Hom}(K_{ab}, \mathbf{U}(1)). \tag{4.1.6}$$

Since K_{ab} and $\mathbf{U}(1)$ are connected, we have an injection from $\mathrm{Hom}(K_{ab}, \mathbf{U}(1))$ to the derivatives:

$$\mathrm{Hom}(\mathfrak{z}(\mathfrak{k}), \mathfrak{u}(1)) = \mathrm{Hom}_{\mathbb{R}}(\mathbf{i}\mathfrak{z}(\mathfrak{g})_{\mathbb{R}}, \mathbf{i}\mathbb{R}) \simeq \mathrm{Hom}_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})_{\mathbb{R}}, \mathbb{R}) = \mathfrak{z}(\mathfrak{g})_{\mathbb{R}}^{\vee}. \tag{4.1.7}$$

Thus, $\mathbf{X}(G)$ is isomorphic to the full-rank lattice $\Theta^{\vee} = \{f \in \mathfrak{z}(\mathfrak{g})_{\mathbb{R}}^{\vee} | f(\mathbf{i}\ker(\exp_{K_{ab}})) \subseteq \mathrm{span}_{\mathbb{Z}}(2\pi)\}$ of $\mathfrak{z}(\mathfrak{g})_{\mathbb{R}}^{\vee} \simeq \mathbb{R}^m$.

(b) The isomorphisms $T \simeq (\mathbb{C}^{\times})^r$ and $G_{ab} \simeq (\mathbb{C}^{\times})^m$ induce isomorphisms $\mathbf{X}(G) \simeq \pi_1(G_{ab})$ and $\mathfrak{z}(\mathfrak{g})_{\mathbb{R}}^{\vee} \simeq \mathfrak{z}(\mathfrak{g})_{\mathbb{R}}$. Thus, $\pi_1(G_{ab})$ embeds into the lattice $\Theta^{\vee}$ of $\mathfrak{z}(\mathfrak{g})_{\mathbb{R}}^{\vee}$, isomorphic to a lattice Θ of $\mathfrak{z}(\mathfrak{g})_{\mathbb{R}}$.

Through the projection $T \to G_{ab}$, we can induce a map $\mathfrak{t}_{\mathbb{R}} \to \mathfrak{z}(\mathfrak{g})_{\mathbb{R}}$, and through the isomorphism $G_{ab} \simeq \mathbf{R}(G)$, we can induce a map $\mathfrak{z}(\mathfrak{g})_{\mathbb{R}} \to \mathfrak{t}_{\mathbb{R}}$. We compose these maps as $\varphi : \mathfrak{z}(\mathfrak{g})_{\mathbb{R}} \to \mathfrak{z}(\mathfrak{g})_{\mathbb{R}}$, which is an isomorphism. From this, we induce an embedding of $\pi_1(G_{ab})$ into $\mathfrak{z}(\mathfrak{g})_{\mathbb{R}}$, as $\Psi = \varphi^{-1}(\Theta)$.

In order to define Harder-Narasimhan types as elements of lattices in $\mathfrak{t}_{\mathbb{R}}$, we need to encode properties of canonical reductions within $\mathfrak{t}_{\mathbb{R}}$. For this, we use Weyl chambers. We now fix a Borel subgroup B of G, containing T, inducing simple roots $\triangle$.

DEFINITION 4.1.10. The set $\overline{C_B} = \{X \in \mathfrak{t}_{\mathbb{R}} | \forall \alpha \in \triangle : \alpha(X) \geq 0\}$ is the *closed Weyl chamber* of B. For brevity, we refer to closed Weyl chambers as Weyl chambers.

REMARK 4.1.11. For a standard parabolic subgroup $P_I \simeq U_I \rtimes L_I$ of G, we have that T is a Cartan subgroup of L_I, which is a complex reductive group, and that $L_I \cap B$ is a Borel subgroup of L_I. Using Lemma 4.1.8, we have a dual-coroot lattice $\Lambda_I = \mathrm{span}_{\mathbb{Z}}(\Phi_H^*(\mathfrak{l}_I, \mathfrak{t}))$ of $\mathfrak{t}_{\mathbb{R}} \simeq \mathbb{R}^r$, such that $\pi_1(L_I) \simeq \Gamma/\Lambda_I$. We obtain the short exact sequence:

$$1 \to \pi_1((L_I)_{der}) \to \pi_1(L_I) \to \pi_1((L_I)_{ab}) \to 1. \tag{4.1.8}$$

Since $\mathbf{R}(G) \subseteq \mathbf{R}(L_I)$, we have $G_{ab} \simeq (\mathbb{C}^{\times})^m$, $(L_I)_{ab} \simeq (\mathbb{C}^{\times})^{m'}$, for $0 \leq m \leq m' \leq r$, such that $\pi_1((L_I)_{ab})$ embeds into $\mathfrak{z}(\mathfrak{l}_I)_{\mathbb{R}} \simeq \mathbb{R}^{m'}$ as a full-rank lattice Ψ_I.

These abstract constructions will be made much clearer through examples, where the real forms, fundamental groups, and lattices, are made explicit.

EXAMPLE 4.1.12. For $G = \mathbf{GL}(r, \mathbb{C})$, we use the notation of Subsection 1.2.1, with the isomorphism $T \simeq (\mathbb{C}^{\times})^r$, mapping every diagonal entry of a matrix to the corresponding r-th component. Thus, we have $\pi_1(T) \simeq \mathbb{Z}^r$.

This induces the real form $K = \mathbf{U}(r)$ of unitary matrices, with the Lie algebra $\mathfrak{u}(r)$. Then $H = K \cap T$ consists of diagonal matrices, whose entries are in $\mathbf{U}(1)$, hence $H \simeq \mathbf{U}(1)^r$. From this, we calculate the following real vector spaces, and the Weyl chamber:

$$\mathfrak{gl}(r, \mathbb{C})_{\mathbb{R}} = \mathbf{i}\mathfrak{u}(r) = \{X \in \mathfrak{gl}(r, \mathbb{C}) | X - X^H = 0\}, \tag{4.1.9}$$

$$\mathfrak{t}_{\mathbb{R}} = \mathfrak{t} \cap \mathfrak{gl}(r, \mathbb{C})_{\mathbb{R}} = \left\{ \begin{pmatrix} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_r \end{pmatrix} \middle| z_1, \ldots, z_r \in \mathbb{R} \right\}, \tag{4.1.10}$$

$$\mathfrak{z}(\mathfrak{gl}(r, \mathbb{C}))_{\mathbb{R}} = \mathfrak{z}(\mathfrak{gl}(r, \mathbb{C})) \cap \mathfrak{gl}(r, \mathbb{C})_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}(I_r), \tag{4.1.11}$$

$$\overline{C_B} = \left\{ \begin{pmatrix} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_r \end{pmatrix} \in \mathfrak{t}_{\mathbb{R}} \middle| z_1 \geq \ldots \geq z_r \right\}. \tag{4.1.12}$$

Furthermore, we use the simple roots $\triangle_H^*$ of $\Phi_H^*(\mathfrak{gl}(r,\mathbb{C}),\mathfrak{t})$ to find the following lattices:

$$\Gamma = \left\{ \begin{pmatrix} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_r \end{pmatrix} \middle| z_1, \ldots, z_r \in \mathbb{Z} \right\}, \tag{4.1.13}$$

$$\Lambda = \left\{ \begin{pmatrix} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_r \end{pmatrix} \in \Gamma \middle| \sum_{i=1}^{n} z_i = 0 \right\}, \qquad \widehat{\Lambda} = \Lambda, \tag{4.1.14}$$

$$\Theta^\vee = \{ f \in \mathfrak{z}(\mathfrak{gl}(r,\mathbb{C}))_{\mathbb{R}}^\vee | f(\mathrm{span}_{\mathbb{Z}}(2\pi I_r)) \subseteq \mathrm{span}_{\mathbb{Z}}(2\pi) \}, \tag{4.1.15}$$

$$\Theta = \mathrm{span}_{\mathbb{Z}}(I_r), \tag{4.1.16}$$

$$\Psi = (1/r)\mathrm{span}_{\mathbb{Z}}(I_r). \tag{4.1.17}$$

From this, we conclude that $\pi_1(\mathbf{GL}(r,\mathbb{C})_{der}) \simeq 1$, $\pi_1(\mathbf{GL}(r,\mathbb{C})) \simeq \mathbb{Z}$ and $\pi_1(\mathbf{GL}(r,\mathbb{C})_{ab}) \simeq \mathbb{Z}$.

We explain how these constructions appear for Levi-factors. We restrict to maximal standard parabolic subgroups $P_I \simeq U_I \rtimes L_I$ of $\mathbf{GL}(r,\mathbb{C})$, where $I = \{\alpha_{s,s+1}\} \subseteq \triangle$. We calculate the following real vector spaces and lattices:

$$\mathfrak{z}(\mathfrak{l}_I)_{\mathbb{R}} = \mathfrak{z}(\mathfrak{l}_I) \cap (\mathfrak{l}_I)_{\mathbb{R}} = \left\{ \begin{pmatrix} z_1 I_s & 0 \\ 0 & z_2 I_{r-s} \end{pmatrix} \middle| z_1, z_2 \in \mathbb{R} \right\}, \tag{4.1.18}$$

$$\Lambda_I = \left\{ \begin{pmatrix} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_r \end{pmatrix} \in \Gamma \middle| \sum_{i=1}^{s} z_i = \sum_{i=s+1}^{r} z_i = 0 \right\}, \qquad \widehat{\Lambda}_I = \Lambda_I, \tag{4.1.19}$$

$$\Theta_I^\vee = \left\{ f \in \mathfrak{z}(\mathfrak{l}_I)_{\mathbb{R}}^\vee \middle| z_1, z_2 \in \mathbb{Z} : f\left(2\pi \begin{pmatrix} z_1 I_s & 0 \\ 0 & z_2 I_{r-s} \end{pmatrix}\right) \subseteq \mathrm{span}_{\mathbb{Z}}(2\pi) \right\}, \tag{4.1.20}$$

$$\Theta_I = \left\{ \begin{pmatrix} z_1 I_s & 0 \\ 0 & z_2 I_{r-s} \end{pmatrix} \middle| z_1, z_2 \in \mathbb{Z} \right\}, \tag{4.1.21}$$

$$\Psi_I = \left\{ \begin{pmatrix} (z_1/s) I_s & 0 \\ 0 & (z_2/(r-s)) I_{r-s} \end{pmatrix} \middle| z_1, z_2 \in \mathbb{Z} \right\}. \tag{4.1.22}$$

From this, we conclude that $\pi_1((L_I)_{der}) \simeq 1$, $\pi_1(L_I) \simeq \mathbb{Z}^2$ and $\pi_1((L_I)_{ab}) \simeq \mathbb{Z}^2$.

These results can obviously be generalized for all standard parabolics P_I and their Levi-factors L_I, which may have more than two diagonal blocks.

EXAMPLE 4.1.13. Using the notation of Subsection 1.2.2, for $G = \overline{\mathbf{SO}}(r,\mathbb{C})$, $r \geq 3$, we write $r = 2n+1$ if r is odd, and $r = 2n$ if r is even. We have the isomorphism $\overline{T} \simeq (\mathbb{C}^\times)^n$, defined by $S_{s_1,\ldots,s_n} \mapsto (s_1, \ldots, s_n)$. Thus, we have $\pi_1(\overline{T}) \simeq \mathbb{Z}^n$.

This induces the real form $K = \overline{\mathbf{SO}}(r,\mathbb{C}) \cap \mathbf{U}(r)$, with the Lie algebra $\mathfrak{k} = \overline{\mathfrak{so}}(r,\mathbb{C}) \cap \mathfrak{u}(r)$. Then $H = K \cap \overline{T}$ consists of diagonal matrices in K, whose entries are in $\mathbf{U}(1)$, hence $H \simeq \mathbf{U}(1)^n$. From this, we calculate the following real vector spaces, and the Weyl chamber:

$$\overline{\mathfrak{so}}(r,\mathbb{C})_{\mathbb{R}} = \mathbf{i}\mathfrak{k} = \{ X \in \mathfrak{gl}(r,\mathbb{C}) | K_r X + X^T K_r = 0, X - X^H = 0 \}, \tag{4.1.23}$$

$$\bar{\mathfrak{t}}_{\mathbb{R}} = \bar{\mathfrak{t}} \cap \overline{\mathfrak{so}}(r,\mathbb{C})_{\mathbb{R}} = \{ Z_{z_1,\ldots z_n} | z_1, \ldots, z_n \in \mathbb{R} \}, \tag{4.1.24}$$

$$\mathfrak{z}(\overline{\mathfrak{so}}(r,\mathbb{C}))_{\mathbb{R}} = 0, \tag{4.1.25}$$

$$\overline{C_{\overline{B}}} = \{ Z_{z_1,\ldots z_n} \in \bar{\mathfrak{t}}_{\mathbb{R}} | z_1 \geq \ldots \geq z_n \geq 0 \}, \text{ if } r \text{ is odd}, \tag{4.1.26}$$

$$\overline{C_{\overline{B}}} = \left\{ Z_{z_1,\dots z_n} \in \overline{\mathfrak{t}}_{\mathbb{R}} | z_1 \geq \ldots \geq z_{n-1} \geq |z_n| \right\}, \text{ if } r \text{ is even.} \tag{4.1.27}$$

Furthermore, we use the simple roots $\triangle_H^*$ of $\Phi_H^*(\overline{\mathfrak{so}}(r,\mathbb{C}),\overline{\mathfrak{t}})$ to find the following lattices:

$$\Gamma = \left\{ Z_{z_1,\dots z_n} | z_1,\dots,z_n \in \mathbb{Z} \right\}, \tag{4.1.28}$$

$$\Lambda = \left\{ Z_{z_1,\dots z_n} \in \Gamma \middle| \sum_{i=1}^{n} z_i \in \mathrm{span}_{\mathbb{Z}}(2) \right\}, \qquad \widehat{\Lambda} = \Gamma, \tag{4.1.29}$$

$$\Theta^\vee = 0, \qquad \Theta = 0, \qquad \Psi = 0. \tag{4.1.30}$$

From this, we conclude that $\pi_1(\overline{\mathbf{SO}}(r,\mathbb{C})_{der}) \simeq \mathbb{Z}/2\mathbb{Z}$, $\pi_1(\overline{\mathbf{SO}}(r,\mathbb{C})) \simeq \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(\overline{\mathbf{SO}}(r,\mathbb{C})_{ab}) \simeq 1$.

We explain how these constructions appear for Levi-factors. We restrict to maximal standard parabolic subgroups $\overline{P}_I \simeq \overline{U}_I \rtimes \overline{L}_I$ of $\overline{\mathbf{SO}}(r,\mathbb{C})$, where $I = \{e_s - e_{s+1}\}, \{e_s\}, \{e_{s-1} + e_s\}$. We know that $\overline{L}_I$ consists of diagonal block matrices of the form:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & K_s(a^{-1})^T K_s \end{pmatrix}, \tag{4.1.31}$$

with $a \in \mathbf{GL}(s,\mathbb{C})$, $d \in \overline{\mathbf{SO}}(r-2s,\mathbb{C})$. We calculate the following real vector spaces and lattices:

$$\mathfrak{z}(\overline{\mathfrak{l}}_I)_{\mathbb{R}} = \mathfrak{z}(\overline{\mathfrak{l}}_I) \cap (\overline{\mathfrak{l}}_I)_{\mathbb{R}} = \left\{ \begin{pmatrix} zI_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -zI_s \end{pmatrix} \middle| z \in \mathbb{R} \right\}, \tag{4.1.32}$$

$$\Lambda_I = \left\{ Z_{z_1,\dots z_n} \in \Gamma \middle| \sum_{i=1}^{s} z_i = 0, \sum_{i=s+1}^{n} z_i \in \mathrm{span}_{\mathbb{Z}}(2) \right\}, \tag{4.1.33}$$

$$\hat{\Lambda}_I = \left\{ Z_{z_1,\dots z_n} \in \Gamma \middle| \sum_{i=1}^{s} z_i = 0 \right\}, \tag{4.1.34}$$

$$\Theta_I^\vee = \left\{ f \in \mathfrak{z}(\mathfrak{l}_I)_{\mathbb{R}}^\vee \middle| z \in \mathbb{Z} : f\left(2\pi \begin{pmatrix} zI_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -zI_s \end{pmatrix}\right) \subseteq \mathrm{span}_{\mathbb{Z}}(2\pi) \right\}, \tag{4.1.35}$$

$$\Theta_I = \left\{ \begin{pmatrix} zI_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -zI_s \end{pmatrix} \middle| z \in \mathbb{Z} \right\}, \tag{4.1.36}$$

$$\Psi_I = \left\{ \begin{pmatrix} (z/s)I_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (-z/s)I_s \end{pmatrix} \middle| z \in \mathbb{Z} \right\}. \tag{4.1.37}$$

From this, we conclude that $\pi_1((\overline{L}_I)_{der}) \simeq \mathbb{Z}/2\mathbb{Z}$, $\pi_1(\overline{L}_I) \simeq \mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, and $\pi_1((\overline{L}_I)_{ab}) \simeq \mathbb{Z}$.

These results can obviously be generalized for all standard parabolics of $\overline{\mathbf{SO}}(r,\mathbb{C})$, and Levi-factors, which may have more diagonal blocks.

Example 4.1.14. Using the notation of Subsection 1.2.3, for $G = \overline{\mathbf{Sp}}(2n,\mathbb{C})$, we have the isomorphism $\overline{T} \simeq (\mathbb{C}^\times)^n$, defined by $S_{s_1,\dots,s_n} \mapsto (s_1,\dots,s_n)$. Thus, we have $\pi_1(\overline{T}) \simeq \mathbb{Z}^n$. This induces the real form $K = \overline{\mathbf{Sp}}(2n,\mathbb{C}) \cap \mathbf{U}(2n)$. From this, we calculate the following real vector spaces, and the Weyl chamber:

$$\overline{\mathfrak{sp}}(2n,\mathbb{C})_{\mathbb{R}} = \mathbf{i}\mathfrak{k} = \{X \in \mathfrak{gl}(r,\mathbb{C}) | J_{2n}X + X^T J_{2n} = 0, X - X^H = 0\}, \tag{4.1.38}$$

$$\overline{\mathfrak{t}}_{\mathbb{R}} = \overline{\mathfrak{t}} \cap \overline{\mathfrak{sp}}(2n,\mathbb{C})_{\mathbb{R}} = \left\{ Z_{z_1,\dots z_n} | z_1,\dots,z_n \in \mathbb{R} \right\}, \tag{4.1.39}$$

$$\mathfrak{z}(\overline{\mathfrak{sp}}(2n,\mathbb{C}))_{\mathbb{R}} = 0, \tag{4.1.40}$$

$$\overline{C_B} = \{Z_{z_1,\dots z_n} \in \bar{\mathfrak{t}}_{\mathbb{R}} | z_1 \geq \dots \geq z_n \geq 0\}\,. \tag{4.1.41}$$

Furthermore, we use the simple roots $\triangle_H^*$ of $\Phi_H^*(\overline{\mathfrak{sp}}(2n,\mathbb{C}),\bar{\mathfrak{t}})$ to find the following lattices:

$$\Gamma = \{Z_{z_1,\dots z_n} | z_1,\dots,z_n \in \mathbb{Z}\}\,, \tag{4.1.42}$$

$$\Lambda = \Gamma, \qquad \widehat{\Lambda} = \Gamma, \tag{4.1.43}$$

$$\Theta^\vee = 0, \qquad \Theta = 0, \qquad \Psi = 0. \tag{4.1.44}$$

From this, we conclude that $\pi_1(\overline{\mathbf{Sp}}(2n,\mathbb{C})_{der}) \simeq 1$, $\pi_1(\overline{\mathbf{Sp}}(2n,\mathbb{C})) \simeq 1$ and $\pi_1(\overline{\mathbf{Sp}}(2n,\mathbb{C})_{ab}) \simeq 1$.

We explain how these constructions appear for Levi-factors. We restrict to maximal standard parabolic subgroups $\overline{P}_I \simeq \overline{U}_I \rtimes \overline{L}_I$ of $\overline{\mathbf{Sp}}(2n,\mathbb{C})$, where $I = \{e_s - e_{s+1}\}, \{2e_s\}$. We calculate the following real vector spaces and lattices:

$$\mathfrak{z}(\bar{\mathfrak{l}}_I)_{\mathbb{R}} = \mathfrak{z}(\bar{\mathfrak{l}}_I) \cap (\bar{\mathfrak{l}}_I)_{\mathbb{R}} = \left\{ \begin{pmatrix} zI_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -zI_s \end{pmatrix} \middle| z \in \mathbb{R} \right\}, \tag{4.1.45}$$

$$\Lambda_I = \left\{ Z_{z_1,\dots z_n} \in \Gamma \middle| \sum_{i=1}^{s} z_i = 0 \right\}, \qquad \hat{\Lambda}_I = \Lambda_I, \tag{4.1.46}$$

$$\Theta_I^\vee = \left\{ f \in \mathfrak{z}(\mathfrak{l}_I)_{\mathbb{R}}^\vee \middle| z \in \mathbb{Z} : f\left(2\pi \begin{pmatrix} zI_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -zI_s \end{pmatrix}\right) \subseteq \operatorname{span}_{\mathbb{Z}}(2\pi) \right\}, \tag{4.1.47}$$

$$\Theta_I = \left\{ \begin{pmatrix} zI_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -zI_s \end{pmatrix} \middle| z \in \mathbb{Z} \right\}, \tag{4.1.48}$$

$$\Psi_I = \left\{ \begin{pmatrix} (z/s)I_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (-z/s)I_s \end{pmatrix} \middle| z \in \mathbb{Z} \right\}. \tag{4.1.49}$$

From this, we conclude that $\pi_1((\overline{L}_I)_{der}) \simeq 1$, $\pi_1(\overline{L}_I) \simeq \mathbb{Z}$ and $\pi_1((\overline{L}_I)_{ab}) \simeq \mathbb{Z}$.

These results can obviously be generalized for all standard parabolics of $\overline{\mathbf{Sp}}(2n,\mathbb{C})$, and Levi-factors, which may have more diagonal blocks.

4.2. Obstruction classes and Harder-Narasimhan types

Now that we understand how fundamental groups of reductive groups appear, we can use them to define obstruction classes of principal bundles. Afterward, through the embeddings of the fundamental groups as lattices of real vector spaces, we can define Harder-Narasimhan types.

For a universal covering $\widetilde{G}$ of G, which is a simply connected complex Lie group, we have the fibration:

$$1 \to \pi_1(G) \to \widetilde{G} \overset{p}{\to} G \to 1, \tag{4.2.1}$$

which defines a fiber bundle p, and a central extension of groups, as $\pi_1(G)$ is abelian.

We denote the algebraic structure sheaf of X by $\mathbf{O}_X$. By passing (4.2.1) through Čech cohomology, we obtain the maps between sets:

$$\dots \to \check{H}^1(X,\mathbf{O}_X(\widetilde{G})) \overset{p_1}{\to} \check{H}^1(X,\mathbf{O}_X(G)) \overset{o_2}{\to} \check{H}^2(X,\mathbf{O}_X(\pi_1(G))). \tag{4.2.2}$$

By construction, the sets $\check{H}^1(X,\mathbf{O}_X(\_))$ correspond to sets of isomorphism classes of principal bundles. These isomorphism classes can be represented by cocycles of principal bundles, which correspond to 1-cocycles in the sense of Čech cohomology.

As X is a compact connected Riemann surface, the singular cohomologies $H^*(X,\pi_1(G))$ and Čech cohomologies $\check{H}^*(X,\mathbf{O}_X(\pi_1(G)))$ are naturally isomorphic. Due to this, by fixing an orientation on the underlying real manifold of X, Poincaré duality gives a canonical isomorphism $\check{H}^2(X, \mathbf{O}_X(\pi_1(G))) \simeq \pi_1(G)$.

We now investigate the connecting map $o_2 : \check{H}^1(X, \mathbf{O}_X(G)) \to \check{H}^2(X, \mathbf{O}_X(\pi_1(G)))$. For a principal-$G$-bundle ξ, its isomorphism class $[\xi] \in \check{H}^1(X, \mathbf{O}_X(G))$ is represented by the cocycles $(\sigma_{ij})_{i,j\in I}$. For all $i, j \in I$, we use the universal lifting property of covering spaces to lift σ_{ij} to $\widetilde{\sigma_{ij}}$, such that the following diagram commutes:

$$\begin{array}{ccc} & \overset{\widetilde{\sigma_{ij}}}{\nearrow} & \widetilde{G} \\ & & \downarrow p \\ U_i \cap U_j & \xrightarrow{\sigma_{ij}} & G \end{array} \quad . \tag{4.2.3}$$

In general, these lifts are not unique, as σ_{ij} can have two different lifts that vary within the fibers of $p : \widetilde{G} \to G$. Furthermore, we cannot guarantee that the cocycle conditions for $(\widetilde{\sigma_{ij}})_{i,j\in I}$, from Definition 2.1.6, are fulfilled. We can measure the defect of this failure by observing for all $i, j, k \in I$:

$$\sigma_{ijk} = \widetilde{\sigma_{jk}}\widetilde{\sigma_{ik}}^{-1}\widetilde{\sigma_{ij}} : U_i \cap U_j \cap U_k \to \pi_1(G). \tag{4.2.4}$$

This defines a 2-cocycle $(\sigma_{ijk})_{i,j,k\in I}$ representing $o_2([\xi]) \in \check{H}^2(X, \mathbf{O}_X(\pi_1(G)))$. If $o_2([\xi]) = 1 \in \check{H}^2(X, \mathbf{O}_X(\pi_1(G)))$, then $[\xi]$ is equivalently the image of an element $[\widetilde{\xi}] \in \check{H}^1(X, \mathbf{O}_X(\widetilde{G}))$, i.e., there exists a principal-$\widetilde{G}$-bundle $\widetilde{\xi}$ such that $\widetilde{\xi}(G) \simeq \xi$.

Definition 4.2.1. Through the isomorphism $\check{H}^2(X, \mathbf{O}_X(\pi_1(G))) \simeq \pi_1(G)$, the class $o_2([\xi]) \in \check{H}^2(X, \mathbf{O}_X(\pi_1(G)))$ induces an element $o_2(\xi) \in \pi_1(G)$, called the *second obstruction class* of ξ.

Through the surjection $\pi_1(G) \to \pi_1(G_{ab})$ from Lemma 4.1.8, we can further map the obstruction class $o_2(\xi)$ to an element $\overline{o_2}(\xi) \in \pi_1(G_{ab})$. As before, we fix a Cartan subgroup T of G, and an isomorphism $T \simeq (\mathbb{C}^\times)^r$. We can now embed the obstruction class of ξ into $\mathfrak{t}_\mathbb{R}$ to define its topological type.

Definition 4.2.2. The fundamental group $\pi_1(G_{ab})$ embeds as a lattice Ψ of $\mathfrak{z}(\mathfrak{g})_\mathbb{R}$, due to (b) of Remark 4.1.9. Through this, $\overline{o_2}(\xi)$ defines an element $\mu^\xi \in \mathfrak{t}_\mathbb{R}$, called the *type* of ξ.

Remark 4.2.3. For $G = \mathbf{GL}(r, \mathbb{C})$, we have that the type μ^ξ is a matrix $(z/r)I_r$, for $z \in \mathbb{Z}$, due to Example 4.1.12.

It can be shown that $z/r = \mu(E_\xi)$, for the vector bundle E_ξ induced from ξ. To prove this, we could use that $\deg(E_\xi) = \int_X c_1(E_\xi)$, where $c_1(E_\xi)$ denotes the *first Chern class* of E_ξ, as constructed in [Fri98, 2.]. We then need to show that the first Chern class corresponds to the second obstruction class.

The following remark explains that obstruction classes and types are functorial.

Remark 4.2.4. Let $\varphi : G \to G'$ be a complex Lie group homomorphism, where G and G' are connected complex reductive groups. Let $\xi(G')$ be the extension of ξ to G' through φ.

(a) Due to the functoriality of Čech cohomology, the induced morphism $\varphi_* : \pi_1(G) \to \pi_1(G')$ maps $o_2(\xi)$ to $o_2(\xi(G'))$.
(b) Since the abelianization of groups is functorial, the morphism φ induces a morphism $\varphi_{ab} : G_{ab} \to G'_{ab}$, such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(G) & \xrightarrow{\varphi_*} & \pi_1(G') \\ \downarrow & & \downarrow \\ \pi_1(G_{ab}) & \xrightarrow{(\varphi_{ab})_*} & \pi_1(G'_{ab}) \end{array} \quad . \tag{4.2.5}$$

Thus, $(\varphi_{ab})_* : \pi_1(G_{ab}) \to \pi_1(G'_{ab})$ maps $\overline{o_2}(\xi)$ to $\overline{o_2}(\xi(G'))$.

(c) For a Cartan subgroup $T \simeq (\mathbb{C}^\times)^r$ of G, and a Cartan subgroup $T' \simeq (\mathbb{C}^\times)^{r'}$ of G', such that $\varphi(\mathbf{R}(G)) \subseteq \mathbf{R}(G')$ and $\varphi(T) \subseteq T'$, the type is also functorial, i.e., $D\varphi$ maps μ^ξ to $\mu^{\xi(G')}$.

We are now able to state the existence and uniqueness of canonical reductions, as proven in Section 3.2.1 and Section 3.2.2, using types.

THEOREM 4.2.5. *There exists a reduction $\sigma^*\xi$ of ξ to a standard parabolic subgroup $P_I \simeq U_I \rtimes L_I$ of G, unique in the sense of Lemma 3.2.6, such that:*

(i) The extension of $\sigma^*\xi$ to the Levi-factor L_I, denoted by $(\sigma^*\xi)(L_I)$, is Ramanathan-semistable.

(ii) For the type $\mu^{(\sigma^*\xi)(L_I)} \in \mathfrak{t}_\mathbb{R}$, we have $\mu^{(\sigma^*\xi)(L_I)} \in \overline{C_B}$. Furthermore, for all $\alpha \in I$, we have $\alpha(\mu^{(\sigma^*\xi)(L_I)}) > 0$.

This reduction $\sigma^*\xi$ is a canonical reduction of ξ.

PROOF. It suffices to show that the conditions *(i)* and *(ii)* are equivalent to the conditions from Theorem 3.2.18. Clearly *(i)* is equivalent to *(i)* of Theorem 3.2.18.

We prove that *(ii)* is equivalent to *(ii)* from Theorem 3.2.18. Let $\chi : P_I \to \mathbb{C}^\times$ be any character, then:

$$D\chi(\mu^{\sigma^*\xi}) = \mu^{\chi(\sigma^*\xi)} = \deg(\chi(\sigma^*\xi)), \tag{4.2.6}$$

where the first equality follows from Remark 4.2.4, and the second follows from Remark 4.2.3. Similarly, for the restriction $\chi|_{L_I} : L_I \to \mathbb{C}^\times$, we have:

$$D\chi(\mu^{(\sigma^*\xi)(L_I)}) = \mu^{\chi((\sigma^*\xi)(L_I))} = \deg(\chi((\sigma^*\xi)(L_I))). \tag{4.2.7}$$

We follow directly that (4.2.6) and (4.2.7) are equal, since $\deg(\chi(\sigma^*\xi)) = \deg(\chi((\sigma^*\xi)(L_I)))$.

Now assume that $\sigma^*\xi$ fulfills *(ii)* of Theorem 3.2.18. For all $\alpha \in \triangle \setminus I$, we have due to the construction of Levi-factors in (1.1.22), that $\alpha \in \Gamma_I \cap -\Gamma_I$ acts trivially on $\mathfrak{z}(\mathfrak{l}_I)$, hence $\alpha(\mu^{(\sigma^*\xi)(L_I)}) = 0$, since $\mu^{(\sigma^*\xi)(L_I)} \in \mathfrak{z}(\mathfrak{l}_I)_\mathbb{R}$. For $\alpha \in I$, let $\chi_\alpha : P_I \to \mathbb{C}^\times$ be a corresponding character from Lemma 3.2.19. Using (4.2.6) and (4.2.7), we have:

$$D\chi_\alpha(\mu^{(\sigma^*\xi)(L_I)}) = \deg(\chi_\alpha(\sigma^*\xi)) > 0, \tag{4.2.8}$$

which is a positive multiple of $\alpha(\mu^{(\sigma^*\xi)(L_I)})$, hence $\alpha(\mu^{(\sigma^*\xi)(L_I)}) > 0$. Altogether, we have that the type $\mu^{\sigma^*\xi(L_I)}$ has the properties laid out in *(ii)*.

The reverse direction is completely analogous, where we apply the same arguments and calculations in reverse. ☐

DEFINITION 4.2.6. We call $\mu^\xi_{HN} = \mu^{(\sigma^*\xi)(L_I)} \in \mathfrak{t}_\mathbb{R}$ the *Harder-Narasimhan type* of ξ.

The Harder-Narasimhan type characterizes the topological type of any canonical reduction of ξ. Due to the functoriality of types, the Harder-Narasimhan type μ^ξ_{HN} also encodes the topological type μ^ξ of ξ. We will now see examples of the Harder-Narasimhan type μ^ξ_{HN}, and how it stores information about canonical reductions of ξ.

EXAMPLE 4.2.7. Let ξ be a principal-$\mathbf{GL}(r, \mathbb{C})$-bundle, with a canonical reduction $\sigma^*\xi$ of ξ to P_I. We know that $\sigma^*\xi$ corresponds to the Harder-Narasimhan filtration of E_ξ:

$$\mathcal{F}_{E_\xi} : 0 = (E_\xi)_0 \subsetneq \ldots \subsetneq (E_\xi)_l = E_\xi, \tag{4.2.9}$$

with the semistable quotient bundles $(F_\xi)_m = (E_\xi)_m/(E_\xi)_{m-1}$, $m = 1, \ldots, l$, of rank r_m.

By following Example 4.1.12, and by noting that types coincide with slopes, as seen in Remark 4.2.3, the Harder-Narasimhan type μ^ξ_{HN} of ξ appears as:

$$\mu^{HN}_\xi = \begin{pmatrix} \mu((F_\xi)_1)I_{r_1} & 0 & \dots & 0 \\ 0 & \mu(F_{\xi 2})I_{r_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \mu((F_\xi)_k)I_{r_l} \end{pmatrix} \in \mathfrak{t}_{\mathbb{R}}. \tag{4.2.10}$$

From this, we see directly that the slope conditions of the Harder-Narasimhan filtration:

$$\mu((F_\xi)_1) > \ldots > \mu((F_\xi)_l), \tag{4.2.11}$$

coincide with the inequalities given in *(ii)* of Theorem 4.2.5.

Example 4.2.8. Let ξ be a principal-$\mathbf{SO}(r, \mathbb{C})$-bundle, with a canonical reduction $\sigma^*\xi$ of ξ to P_I. We know that $\sigma^*\xi$ induces a special orthogonal Harder-Narasimhan filtration of (E_ξ, β, τ):

$$\mathcal{F}_{(E_\xi,\beta)} : 0 = (E_\xi)_0 \subsetneq \ldots \subsetneq (E_\xi)_l \subsetneq E_\xi, \tag{4.2.12}$$

with semistable quotient bundles $(F_\xi)_m = (E_\xi)_m/(E_\xi)_{m-1}$, $m = 1, \ldots, l$, of rank r_m.

Since the types of isotropic subbundles of E_ξ coincide with slopes, similar to Remark 4.2.3, the Harder-Narasimhan type μ^ξ_{HN} of ξ appears as:

$$\mu^\xi_{HN} = \begin{pmatrix} \mu((F_\xi)_1)I_{r_1} & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mu((F_\xi)_l)I_{r_l} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\mu((F_\xi)_l)I_{r_l} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & -\mu((F_\xi)_1)I_{r_1} \end{pmatrix} \in \bar{\mathfrak{t}}_{\mathbb{R}}, \tag{4.2.13}$$

where the middle 0 column does not appear in the even $r = 2n$ case. From this, the slope conditions on the quotients:

$$\mu((F_\xi)_1) > \ldots > \mu((F_\xi)_l) > 0, \tag{4.2.14}$$

fulfill the inequalities given in *(ii)* of Theorem 4.2.5.

Example 4.2.9. Let ξ be a principal-$\mathbf{Sp}(2n, \mathbb{C})$-bundle, with a canonical reduction $\sigma^*\xi$ of ξ to P_I. We know that $\sigma^*\xi$ induces a symplectic Harder-Narasimhan filtration of (E_ξ, β):

$$\mathcal{F}_{(E_\xi,\beta)} : 0 = (E_\xi)_0 \subsetneq \ldots \subsetneq (E_\xi)_l \subsetneq E_\xi, \tag{4.2.15}$$

with semistable quotient bundles $(F_\xi)_m = (E_\xi)_m/(E_\xi)_{m-1}$, $m = 1, \ldots, l$, of rank r_m.

Since the types of isotropic subbundles of E_ξ coincide with slopes, similar to Remark 4.2.3, the Harder-Narasimhan type μ^ξ_{HN} of ξ appears as:

$$\mu^\xi_{HN} = \begin{pmatrix} \mu((F_\xi)_1)I_{r_1} & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mu((F_\xi)_l)I_{r_l} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\mu((F_\xi)_l)I_{r_l} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 & -\mu((F_\xi)_1)I_{r_1} \end{pmatrix} \in \bar{\mathfrak{t}}_{\mathbb{R}}, \tag{4.2.16}$$

From this, the slope conditions on the quotients:

$$\mu((F_\xi)_1) > \ldots > \mu((F_\xi)_l) > 0, \qquad (4.2.17)$$

coincide with the inequalities given in *(ii)* of Theorem 4.2.5.

References

Books

[Bor91] Armand Borel. *Linear Algebraic Groups*. Springer, 1991. URL: https://link.springer.com/book/10.1007/978-1-4612-0941-6.

[Fri98] Robert Friedman. *Algebraic Surfaces and Holomorphic Vector Bundles*. Springer, 1998. URL: https://link.springer.com/book/10.1007/978-1-4612-1688-9.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry*. John Wiley and Sons, 1994. URL: https://onlinelibrary.wiley.com/doi/book/10.1002/9781118032527.

[Hal15] Brian Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Springer, 2015. URL: https://link.springer.com/book/10.1007/978-3-319-13467-3.

[Ham17] Mark Hamilton. *Mathematical Gauge Theory*. Springer, 2017. URL: https://link.springer.com/book/10.1007/978-3-319-68439-0.

[Hum72] James Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer, 1972. URL: https://www.math.uci.edu/~brusso/humphreys.pdf.

[Kna88] Anthony Knapp. *Lie Groups Beyond an Introduction, Second Edition*. Birkhäuser, 1988. URL: https://www.math.stonybrook.edu/~aknapp/books/green/beyond2-frontmatter.pdf.

[MT12] Gunter Malle and Donna Testerman. *Linear Algebraic Groups and Finite Groups of Lie Type*. Cambridge University Press, 2012. URL: https://www.cambridge.org/core/books/linear-algebraic-groups-and-finite-groups-of-lie-type/DF7451BDFAB4ED163BB88268EF7490BD.

Manuscripts, Lectures, and Papers

[AB82] Michael Atiyah and Raoul Bott. *The Yang-Mills Equations over Riemann Surfaces*. 1982. URL: http://www.math.toronto.edu/mgualt/Morse%20Theory/Atiyah-Bott.pdf.

[BH04] Indranil Biswas and Yogish Holla. *Harder-Narasimhan Reduction of a Principal Bundle*. 2004. URL: https://projecteuclid.org/journals/nagoya-mathematical-journal/volume-174/issue-none/Harder-Narasimhan-reduction-of-a-principal-bundle/nmj/1114632072.full.

[Čap23] Andreas Čap. *Lie Groups*. 2023. URL: https://www.mat.univie.ac.at/~cap/files/LieGroups.pdf.

[CE48] Claude Chevalley and Samuel Eilenberg. *Cohomology Theory of Lie Groups and Lie Algebras*. 1948. URL: https://www.ams.org/journals/tran/1948-063-01/S0002-9947-1948-0024908-8/S0002-9947-1948-0024908-8.pdf.

[Don83] Simon Donaldson. *A New Proof of a Theorem of Narasimhan and Seshadri*. 1983. URL: https://projecteuclid.org/journals/journal-of-differential-geometry/volume-18/issue-2/A-new-proof-of-a-theorem-of-Narasimhan-and-Seshadri/10.4310/jdg/1214437664.full.

[HM04] Donghoon Hyeon and David Murphy. *Note on the Stability of Principal Bundles*. 2004. URL: https://www.ams.org/journals/proc/2004-132-08/S0002-9939-04-07386-1/S0002-9939-04-07386-1.pdf.

[HN75] Günter Harder and M. S. Narasimhan. *On the Cohomology Groups of Moduli Spaces of Vector Bundles on Curves*. 1975. URL: https://link.springer.com/content/pdf/10.1007/BF01357141.pdf.

[Mum62] David Mumford. *Projective Invariants of Projective Structures and Applications*. 1962. URL: https://www.dam.brown.edu/people/mumford/alg_geom/papers/1962c--ICMStockholm-IMU.pdf.

[NS65] M. S. Narasimhan and C. S. Seshadri. *Stable and Unitary Vector Bundles on a Compact Riemann Surface*. 1965. URL: https://www.jstor.org/stable/1970710.

[Ram75] A. Ramanathan. *Stable Principal Bundles on a Compact Riemann Surface*. 1975. URL: https://link.springer.com/article/10.1007/bf01343949.

[RR83] S. Ramanan and A. Ramanathan. *Some Remarks on the Instability Flag*. 1983. URL: https://projecteuclid.org/journals/tohoku-mathematical-journal/volume-36/issue-2/Some-remarks-on-the-instability-flag/10.2748/tmj/1178228852.full.