Faculty of Mathematics and Computer Science

Heidelberg University

Bachelor Thesis in Mathematics submitted by:

Emanuel Roth

Matriculation Number: 4013284

Birthplace: Jerudong, Brunei

2021

The Mitchell Embedding Theorem

Mitchells Einbettungssatz

Zusammenfassung:

Diese Bachelorarbeit konstruiert die erforderlichen Elementen der Kategorientheorie, um *Mitchells Einbettungssatz* darzustellen, zu erklären und zu beweisen. Der Satz besagt, dass jede kleine abelsche Kategorie eine volltreue exakte Einbettung in eine Modulkategorie besitzt, was uns im Wesentlichen erlaubt, kleine abelsche Kategorien als Unterkategorien von Modulkategorien zu interpretieren.

Dafür stellen wir bekannte Einbettungskonstruktionen und deren Eigenschaften vor, wie die Yoneda Einbettung mit ihren Faktorisierungen zusammen mit dem *Yoneda Lemma*.

Nachdem wir *Mitchells Einbettungssatz* bewiesen haben, erklären wir einige Konsequenzen und Folgerungen dieses Satzes und untersuchen, wie der Satz angewendet werden kann, um nützliche Ergebnisse aus der homologischen Algebra, sowie das Fünferlemma und das Schlangenlemma, zu verallgemeinern. Diese Verallgemeinerungen nehmen typerscherweise Aussagen, die für Kategorien von Modulen gelten, und erweitern sie zu analogen, stärkeren Aussagen, die für alle abelschen Kategorien gelten.

Abstract:

This bachelor thesis constructs the required category theory to adequately present, explain and prove the *Mitchell embedding theorem*. The theorem states that every small abelian category has a fully faithful exact embedding into a category of modules, which essentially allows us to interpret small abelian categories as subcategories of categories of modules.

To do this, we introduce and develop well-known embedding constructions and their properties, such as the Yoneda embedding, and its factorizations, along with the *Yoneda lemma*.

After proving the *Mitchell embedding theorem*, we explain some consequences and corollaries of this theorem, and explore how to apply the theorem to generalize useful results from homological algebra, including the five lemma and the snake lemma. These generalizations usually take statements which are true for categories of modules and extend them to analogous, more powerful statements which are true for all abelian categories.

Diese Bachelorarbeit wurde von Emanuel Roth an der Fakultät für Mathematik und Informatik in Heidelberg unter der Betreuung von Prof. Dr. Gebhard Böckle erstellt

Contents

0 Motivation and Introduction

0.1 Motivation and Introduction

Category theory was introduced in the 1940s by *Saunders Mac Lane* and *Samuel Eilenberg* to serve as a framework where homology and homotopy could be studied in a more algebraic context than prior attempts. Category theory's strength is that it identifies similar properties between wide-ranging constructions and gives us a deep insight on their behavior and structure without having to deal with explicit pedestrian examples and arguments. It can be applied to groups, fields, modules, Banach spaces, topologies, and many more constructions.

Homological algebra, in its simplest form, first arose as the study of homology and homotopy within topological spaces. Slowly the field expanded and generalized with the introduction of category theory and commutative algebra. The field's more generalized statements and theorems have wide-ranging applications in algebraic geometry and topology, algebraic number theory, complex analysis and many more disciplines.

The *Mitchell embedding theorem* (**III.3.3.1** [Mor20]), ([NCa19]) is a useful result in homological algebra, which was proven by *Peter Freyd* and *Barry Mitchell* in the 1960s. It gives us an additional structure to certain abelian categories that lets us identify such categories with subcategories of modules. This theorem has several applications that help us transfer known results from categories of modules, which are typically well-understood, to abelian categories, which have less structure and are more generalized. This is often rather useful because abelian categories are frequently used constructions in homological algebra. The theorem can be applied to help us better understand certain abelian categories of sheaves within algebraic geometry and sheaf theory. Many so-called "diagram chasing" theorems for modules from homological algebra, such as the five lemma, snake lemma, nine lemma and more, can be generalized to work for abelian categories rather easily using the *Mitchell embedding theorem*.

This thesis mainly follows the first three chapters of *Sophie Morel* in *Homological Algebra* ([Mor20]), and partially *Martin Brandenburg* in *Einführung in die Kategorientheorie* ([Bra16]). Where some details may be expanded upon or left out according to their general usefulness or relevance to proving the *Mitchell embedding theorem*.

0.2 Structure of the Thesis

The thesis will be structured as follows: **Section 1** introduces the basics of category theory with some examples and **Section 2** introduces important types of categories such as additive and abelian categories, as well as many useful related statements and lemmas. **Section 3** explains the dual concepts of injectives and projectives, as well as generators and cogenerators. Injectives and projectives fulfill certain universal properties that have nice properties in categories with generators and cogenerators. In particular, we introduce Grothendieck abelian categories here. **Section 4** gives a brief introduction to sheaves and in particular sheaves on Grothendieck pretopologies. This section also constructs and discusses sheafification and its useful properties. **Section 5** puts all our results together to prove *Mitchell's embedding theorem* and afterwards explains some useful applications and consequences of the theorem.

The first two sections of this thesis may be prior knowledge and thus may be skipped or partially skipped. Material more directly relevant to proving *Mitchell's embedding theorem* begins nearing the end of **Section 2**.

1 Category-Theoretic Preliminaries

1.1 Grothendieck Universes

1.1.1 Motivation (Grothendieck Universes) (I.1 [Mor20]**):** We will work under *Zermelo-Fraenkel set theory with the axiom of choice* (written briefly as *ZFC*) ([Wik21e]). We want to introduce some important and general definitions, such as graphs and categories, which contain objects and morphisms. For example, we want to be able to talk about a graph or category whose objects are sets.

A naïve attempt to formalize this idea would be to define a graph $\mathcal{G}$ as containing the information of a set of objects $\mathrm{Ob}(\mathcal{G})$ and sets of morphisms $\mathrm{Hom}_{\mathcal{G}}(A, B)$ for all objects A and B in $\mathcal{G}$. However formally this becomes problematic when we discuss the graph of sets. Here the objects would form the set of all sets, which is not a well-defined set within *ZFC*, this can be shown through *Russell's paradox* ([Mat20]), or shown with an argument involving cardinality, known thanks to *Georg Cantor* ([Mat21]).

There are multiple solutions to help resolve this which mainly involve extending *ZFC* sensibly. One may use *von Neumann-Bernays-Gödel set theory* ([Wik21d]) which introduces the notion of classes, which are more general than sets. However, *Mitchell's embedding theorem* wouldn't work for some categories defined through this approach. We instead use Grothendieck's extension of *ZFC* which introduces Grothendieck universes to avoid working with classes. In our graph of sets example, we want to "shrink" the graph of sets so that not all sets are objects.

1.1.2 Definition (Grothendieck Universes) (I.1.1 [Mor20]**):** A *Grothendieck universe* $\mathcal{U}$ is a set containing sets as elements with the following properties:

(i) $\emptyset \in \mathcal{U}$.

(ii) If $x \in \mathcal{U}$ and $y \in x$, then $y \in \mathcal{U}$.

(iii) If $x \in \mathcal{U}$, then $\{x\} \in \mathcal{U}$.

(iv) If $x \in \mathcal{U}$, then $\mathcal{P}(x) \in \mathcal{U}$, whereby $\mathcal{P}(x)$ is the power set of x.

(v) If $(x_i)_{i \in I}$ is a family of sets in $\mathcal{U}$ indexed by $I \in \mathcal{U}$, we have $\bigcup_{i \in I} x_i \in \mathcal{U}$.

(vi) The natural numbers are in $\mathcal{U}$, i.e. $\mathbb{N} \in \mathcal{U}$.

1.1.3 Notes (Grothendieck Universes):

(a) (**1.1.2**(ii)) and (**1.1.2**(iv)) together imply (**1.1.2**(iii)), because for $x \in \mathcal{U}$ we have $\{x\} \in \mathcal{P}(x) \in \mathcal{U}$ and therefore $\{x\} \in \mathcal{U}$.

(b) Here we note that (**1.1.2**(vi)) and (**1.1.2**(ii)) would imply that natural numbers, such as $3 \in \mathcal{U}$ are sets themselves. To make sense of this, we must view natural numbers set-theoretically and define $0 = \emptyset$, $n + 1 = \mathcal{P}(n)$ recursively for all n.

(c) Applying (**1.1.2**(iv)) and (**1.1.2**(ii)) gives the following statement: If $x \in \mathcal{U}$ and $y \subset x$, then we have $y \in \mathcal{U}$.

(d) Grothendieck universes, given they exist, are sufficiently large enough and contain enough cardinalities for us to sensibly to define categories and graphs using them whilst avoiding set-theoretic problems, such as *Cantor's theorem* ([Mat21]) and *Russell's paradox* ([Mat20]).

(e) Under *ZFC*, the sets within a Grothendieck universe also fulfill *ZFC*, as universes are sets.

1.1.4 Tarski-Grothendieck Set Theory ([Wik21c]**):** *Tarski-Grothendieck set theory* is an extension of *ZFC* which contains *ZFC* and the following additional axiom (a statement independent of *ZFC*):

Axiom of universes: Every set lies within a Grothendieck universe.

This is useful as it enables us to construct larger and larger universes always containing specific sets.

For the rest of the thesis, we will work with *Tarski-Grothendieck set theory*.

1.1.5 Definitions ($\mathcal{U}$-Sets and $\mathcal{U}$-Small Sets) (I.2.1.3 [Mor20]**):** For a Grothendieck universe $\mathcal{U}$ we define:

(a) A set x is a $\mathcal{U}$*-set* if $x \in \mathcal{U}$.

(b) A set x is $\mathcal{U}$*-small* if there exists a $y \in \mathcal{U}$ and a bijection between x and y.

1.2 Categories

With Grothendieck universes we can now define the categories and graphs that we will need throughout the thesis. From now on let $\mathcal{U}$ be a Grothendieck universe.

1.2.1 Definition (Graphs) (2.5 [Bö20]**):** A *graph* $\mathcal{G} = (\mathrm{Ob}(\mathcal{G}), \mathrm{Mor}(\mathcal{G}), \mathrm{dom}, \mathrm{cod})$ consists of a tuple consisting of a set of objects $\mathrm{Ob}(\mathcal{G})$ and morphisms, denoted by $\mathrm{Mor}(\mathcal{G})$ (not necessarily a set), such that we have the mappings $\mathrm{dom} : \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$ and $\mathrm{cod} : \mathrm{Mor}(\mathcal{G}) \to \mathrm{Ob}(\mathcal{G})$.

For a morphism f in $\mathrm{Mor}(\mathcal{G})$, the *domain* of f is $\mathrm{dom}(f) = A$ and the *codomain* of f is $\mathrm{cod}(f) = B$. We can more briefly write $f : A \to B$ and call f a *morphism from* A *to* B.

1.2.2 Definition (Collections of Morphisms) (2.9 [Bö20]**):** For all objects A and B in a graph $\mathcal{G}$, the *morphisms from* A *to* B are denoted by $\mathrm{Hom}_{\mathcal{D}}(A, B)$ (not necessarily a set).

1.2.3 Definition (Categories) (2.7, 2.8 [Bö20]**):** A *category* $\mathcal{C} = (\mathrm{Ob}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}), \mathrm{dom}, \mathrm{cod}, 1, \circ)$ consists of a graph $(\mathrm{Ob}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}), \mathrm{dom}, \mathrm{cod})$ with:

(i) A mapping $1 : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C}), A \mapsto (1_A : A \to A)$ that maps to *identities* of objects.

(ii) A mapping $\circ : (\mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor})(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C}), (g, f) \mapsto g \circ f$ whereby:

$$(\mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor})(\mathcal{C}) = \{(g, f) \in \mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C}) | \mathrm{dom}(g) = \mathrm{cod}(f)\},$$

denotes pairs of morphisms whereby *composition* is possible.

The composition $\circ$ and identity 1 must also have the following properties:

(a) For all pairs (g, f) such that composition is possible with $f : A \to B$ and $g : B \to C$, we have a morphism $g \circ f : A \to C$ in $\mathrm{Mor}(\mathcal{C})$.

(b) For all morphisms $f : A \to B$ we have $f = f \circ 1_A = 1_B \circ f$.

(c) $\circ$ is an associative composition, meaning that for all pairs (h, g) and (g, f) with composition we have $h \circ (g \circ f) = (h \circ g) \circ f$.

1.2.4 Definitions ($\mathcal{U}$-Categories and $\mathcal{U}$-Small Categories) (I.2.1.3 [Mor20]**):** For a Grothendieck universe $\mathcal{U}$ and a category $\mathcal{C}$:

(a) $\mathcal{C}$ is a $\mathcal{U}$*-category* if for all objects A and B it follows that $\mathrm{Hom}_{\mathcal{C}}(A, B) \in \mathcal{U}$.

(b) $\mathcal{C}$ is a $\mathcal{U}$*-small* if $\mathcal{C}$ is a $\mathcal{U}$-category and additionally $\mathrm{Ob}(\mathcal{C})$ is $\mathcal{U}$-small.

1.2.5 Examples (Categories) (I.2.1.7 [Mor20]**):**

(a) $\mathrm{SET}_{\mathcal{U}}$ is the *category of* $\mathcal{U}$*-sets*, where the objects are $\mathcal{U}$-sets, i.e. $\mathrm{Ob}(\mathrm{SET}_{\mathcal{U}}) = \mathcal{U}$ and the morphisms are functions between $\mathcal{U}$-sets. It is obvious that the composition of functions works as it should, and that it is associative. Furthermore, using $1(A) = \mathrm{id}_A$ as the identity function for all objects works as it should.

For two objects A and B in $\mathrm{SET}_{\mathcal{U}}$, $\mathrm{Hom}_{\mathrm{SET}_{\mathcal{U}}}(A, B)$ is a $\mathcal{U}$-set. One can show that $A, B \in \mathcal{U}$ implies $A \times B \in \mathcal{U}$ (**I.1.3**(ii) [Mor20]) and therefore $\mathcal{P}(A \times B) \in \mathcal{U}$. Functions $f : A \to B$ can be encoded as a particular subset of $A \times B$ by identifying f with $\{(a, f(a)) | a \in A\}$. Thus $\mathrm{Hom}_{\mathrm{SET}_{\mathcal{U}}}(A, B)$, as the collection of all such sets $\{(a, f(a)) | a \in A\}$, is a subset of $\mathcal{P}(A \times B) \in \mathcal{U}$ i.e. $\mathrm{Hom}_{\mathrm{SET}_{\mathcal{U}}}(A, B) \in \mathcal{P}(\mathcal{P}(A \times B)) \in \mathcal{U}$ and thus $\mathrm{Hom}_{\mathrm{SET}_{\mathcal{U}}}(A, B) \in \mathcal{U}$.

With this statement, $\mathrm{SET}_{\mathcal{U}}$ is a $\mathcal{U}$-category, but not a $\mathcal{U}$-small category as $\mathrm{Ob}(\mathrm{SET}_{\mathcal{U}}) = \mathcal{U}$ is not a $\mathcal{U}$-small set due to (**I.1.7**(i) [Mor20]).

(b) $\text{AB}_{\mathcal{U}}$ is the *category of abelian $\mathcal{U}$-groups*, whereby its objects are sets $G \in \mathcal{U}$ with an abelian group structure $(G, +, e)$, and its morphisms are group homomorphisms. Similar to $\text{SET}_{\mathcal{U}}$, $\text{AB}_{\mathcal{U}}$ defines a $\mathcal{U}$-category.

(c) $\text{TOP}_{\mathcal{U}}$ is the *category of $\mathcal{U}$-topologies*, whereby its objects (X, τ) are $\mathcal{U}$-sets X endowed with a topology τ, and its morphisms are continuous functions between topologies. $\text{TOP}_{\mathcal{U}}$ also defines a $\mathcal{U}$-category.

(d) Let R be a $\mathcal{U}$-ring, i.e. a ring such that $R \in \mathcal{U}$. ${}_R\text{MOD}_{\mathcal{U}}$ is the *category of $\mathcal{U}$-left-R-modules*, whereby its objects are left-R-modules $(M, 0, 1, +, \cdot)$ such that $M \in \mathcal{U}$. Morphisms in ${}_R\text{MOD}_{\mathcal{U}}$ are module homomorphisms (i.e. R-linear functions) and ${}_R\text{MOD}_{\mathcal{U}}$ is again a $\mathcal{U}$-category.

We analogously define $\text{MOD}_{R\mathcal{U}}$ as the *category of $\mathcal{U}$-right-R-modules*.

1.2.6 Definition (Dual Categories) (2.18 [Bö20]**):** Dual categories are important as they are used for presheaves and sheaves. Let $\mathcal{C}$ be a category, then its *dual category* is a category $\mathcal{C}^{\text{op}}$ that fulfills the following:

(i) $\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C}^{\text{op}})$, i.e. they have the same objects.

(ii) There exists a bijection $\text{Mor}(\mathcal{C}) \to \text{Mor}(\mathcal{C}^{\text{op}}), f \mapsto f^{\text{op}}$ between the morphisms with the following properties:

  (a) The domains and codomains are swapped, i.e. $\text{dom}(f) = \text{cod}^{\text{op}}(f^{\text{op}})$ and $\text{cod}(f) = \text{dom}^{\text{op}}(f^{\text{op}})$, whereby dom^{op} and cod^{op} are domains and codomains in $\mathcal{C}^{\text{op}}$.

  (b) If we denote $\circ$ as the composition in $\mathcal{C}$ and $\circ^{\text{op}}$ as the composition in $\mathcal{C}$, then we have for f, g and $g \circ f$ in $\text{Mor}(\mathcal{C})$ that $(g \circ f)^{\text{op}} = f^{\text{op}} \circ^{\text{op}} g^{\text{op}}$.

$\mathcal{C}^{\text{op}}$ is also uniquely defined from $\mathcal{C}$ and when it is clear from context, we will write f instead of f^{op}.

1.2.7 Note (Dual Categories): The dual of a dual of a category is itself again, i.e. for a category $\mathcal{C}$ we have $\mathcal{C} = (\mathcal{C}^{\text{op}})^{\text{op}}$.

1.2.8 Definition (Product Categories) (2.26 [Bö20]**):** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, then the *product category* $\mathcal{C} \times \mathcal{D}$ is the category whereby the objects are tuples (C, D) such that $C \in \text{Ob}(\mathcal{C})$ and $D \in \text{Ob}(\mathcal{D})$. The morphisms are exactly the tuples $(f, g) : (C, D) \to (E, F)$ whereby $f : C \to E$ is a morphism in $\mathcal{C}$ and $g : D \to F$ is a morphism in $\mathcal{D}$.

In $\mathcal{C} \times \mathcal{D}$, the pair of morphisms (f, g) and (h, i) can only composed as $(h, i) \circ (f, g) = (h \circ f, i \circ g)$ if and only if $\text{cod}(f) = \text{dom}(h)$ and $\text{cod}(g) = \text{dom}(i)$.

1.2.9 Definitions (Left and Right Compositions) (2.29 [Bö20]**):** Let $\mathcal{C}$ be a category and $f : A \to B$ be a morphism in $\mathcal{C}$.

(a) **Left Compositions:** Let W be an object in $\mathcal{C}$, then we define *left composition* as the function $f_* : \text{Hom}_{\mathcal{C}}(W, A) \to \text{Hom}_{\mathcal{C}}(W, B), g \mapsto f \circ g$.

(b) **Right Compositions:** Let W be an object in $\mathcal{C}$, then we define *right composition* as the function $f^* : \text{Hom}_{\mathcal{C}}(B, W) \to \text{Hom}_{\mathcal{C}}(A, W), h \mapsto h \circ f$.

1.2.10 Definitions (Types of Morphisms) (2.30 [Bö20]**):** Let $\mathcal{C}$ be a category and $f : A \to B$ be a morphism in $\mathcal{C}$.

(a) **Monomorphisms:** f is a *monomorphism* if f_* is injective for all objects W in $\mathcal{C}$.

(b) **Epimorphisms:** f is an *epimorphism* if f^* is injective for all objects W in $\mathcal{C}$.

(c) **Isomorphisms:** f is an *isomorphism* if there exists a morphism $g : B \to A$ in $\mathcal{C}$, such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

1.2.11 Lemma (Dual Categories) (2.31 [Bö20]**),** ([NCa21d])**:** For a category $\mathcal{C}$ and a morphism $f : A \to B$, it follows that:

(a) f is a monomorphism in $\mathcal{C}$ if and only if f is an epimorphism in $\mathcal{C}^{\mathrm{op}}$.

(b) f is an epimorphism in $\mathcal{C}$ if and only if f is a monomorphism in $\mathcal{C}^{\mathrm{op}}$.

Proof: See references for a general explanation. □

1.3 Functors

We've reviewed categories since categories are sometimes defined differently in different contexts. Functors have more standard definitions. Let $\mathcal{U}$ be a Grothendieck universe.

1.3.1 Definition (Functors) (I.2.2.1 [Mor20]**):** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ comprises of:

(i) A function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D}), A \mapsto FA$ (here a function as $\mathrm{Ob}(\mathcal{C})$, $\mathrm{Ob}(\mathcal{D})$ are sets).

(ii) A mapping $F : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D}), (f : A \to B) \mapsto (Ff : FA \to FB)$ such that for two morphisms in $\mathcal{C}$ where a composition $g \circ f$ exists, it follows that $F(g \circ f) = Fg \circ Ff$.

The image of the function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, along with the associated morphisms $(Ff : FA \to FB)_{f \in \mathrm{Mor}(\mathcal{C})}$ form a subgraph $F(\mathcal{C})$ of $\mathcal{D}$, which we call the *image* of F.

1.3.2 Examples (Functors): Important examples of functors include the following:

(a) **Opposite Functors:** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories $\mathcal{C}$ and $\mathcal{D}$, then there exists a canonical *opposite functor* $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ between categories $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{D}^{\mathrm{op}}$ that is uniquely derived from F. Due to $(F^{\mathrm{op}})^{\mathrm{op}} : \mathcal{C} \to \mathcal{D}$ being the same as F, F is also uniquely derived from F^{op}.

(b) **Forgetful Functors:** Forgetful functors describe inclusions of categories into other categories. In general, *forgetful functors* typically map from objects with more structure to the same objects, but with less structure.

For example, let R be an $\mathcal{U}$-ring, we then define the *forgetful functor* $\mathrm{For} : {}_R\textsc{Mod}_{\mathcal{U}} \to \textsc{Set}_{\mathcal{U}}$, which maps $\mathcal{U}$-left-R-modules and R-linear functions to their underlying $\mathcal{U}$-sets and functions. Analogously there also exists a *forgetful functor* $\mathrm{For} : \textsc{Mod}_{R\mathcal{U}} \to \textsc{Set}_{\mathcal{U}}$

$\mathrm{For} : \textsc{Ab}_{\mathcal{U}} \to \textsc{Grp}_{\mathcal{U}}$ is the *forgetful functor* that maps abelian $\mathcal{U}$-groups to $\mathcal{U}$-groups.

(c) **Hom-Functors:** Let $\mathcal{C}$ be a $\mathcal{U}$-category and A be an object in $\mathcal{C}$. We define the mappings:

$$\mathrm{Hom}_{\mathcal{C}}(\_, A) : \mathcal{C}^{\mathrm{op}} \to \textsc{Set}_{\mathcal{U}}, W \mapsto \mathrm{Hom}_{\mathcal{C}}(W, A), (f : W \to V) \mapsto f^*,$$

$$\mathrm{Hom}_{\mathcal{C}}(A, \_) : \mathcal{C} \to \textsc{Set}_{\mathcal{U}}, W \mapsto \mathrm{Hom}_{\mathcal{C}}(A, W), (f : W \to V) \mapsto f_*,$$

with the morphisms $f^* : \mathrm{Hom}_{\mathcal{C}}(V, A) \to \mathrm{Hom}_{\mathcal{C}}(W, A)$ and $f_* : \mathrm{Hom}_{\mathcal{C}}(A, W) \to \mathrm{Hom}_{\mathcal{C}}(A, V)$ from (**1.2.9**(a)) and (**1.2.9**(b)). As $\mathcal{C}$ is a $\mathcal{U}$-category, the sets $\mathrm{Hom}_{\mathcal{C}}(W, A)$ and $\mathrm{Hom}_{\mathcal{C}}(A, W)$ above are $\mathcal{U}$-sets and therefore the mappings above define functors $\mathrm{Hom}_{\mathcal{C}}(\_, A)$ and $\mathrm{Hom}_{\mathcal{C}}(A, \_)$, which are called *hom-functors*.

Furthermore, with the product category $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, we can combine these two functors to create the *hom-functor* $\mathrm{Hom}_{\mathcal{C}}(\_, \_)$:

$$\mathrm{Hom}_{\mathcal{C}}(\_, \_) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \textsc{Set}_{\mathcal{U}}, (A, B) \mapsto \mathrm{Hom}_{\mathcal{C}}(A, B),$$

$$((f, g) : (A, B) \to (C, D)) \mapsto ((f^*, g_*) : \mathrm{Hom}_{\mathcal{C}}(C, B) \to \mathrm{Hom}_{\mathcal{C}}(A, D), h \mapsto g \circ h \circ f),$$

whereby $f : A \to C$, $g : B \to D$ are morphisms in $\mathcal{C}$. Generalizations of these functors will be important for defining adjoint functors in (**1.3.5**(g)).

1.3.3 Definition (Natural Transformations) (I.2.3.1 [Mor20]): Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $u : F \to G$ is a collection of morphisms $(uA : FA \to GA)_{A \in \mathrm{Ob}(\mathcal{C})}$ in $\mathcal{D}$ such that for all morphisms $f : A \to B$ in $\mathcal{C}$, the following diagram in $\mathcal{D}$ commutes:

$$\begin{array}{ccc} FA & \overset{uA}{\to} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \overset{uB}{\to} & GB \end{array} .$$

We then define $\textsc{Func}(\mathcal{C}, \mathcal{D})$ as the *category of functors from* $\mathcal{C}$ *to* $\mathcal{D}$: the objects of $\textsc{Func}(\mathcal{C}, \mathcal{D})$ are functors from $\mathcal{C}$ to $\mathcal{D}$, and the morphisms $u : F \to G$ of $\textsc{Func}(\mathcal{C}, \mathcal{D})$ are natural transformations from F to G.

1.3.4 Notes (Natural Transformations):

(a) It can be shown that a natural transformation $u : F \to G$ in $\textsc{Func}(\mathcal{C}, \mathcal{D})$ is a monomorphism (respectively epimorphism, isomorphism) if for all objects A in $\mathcal{C}$, $uA : FA \to GA$ is a monomorphism, (respectively epimorphism, isomorphism) in $\mathcal{D}$. For isomorphisms, the converse implication is true (**2.50** [Bö20]). If $\mathcal{D}$ has all *pullbacks* (respectively all *pushouts*) as defined later in (**2.2.7**), then the converse implication for monomorphisms (respectively epimorphisms) is true ([Mat14]).

We say there is a *natural isomorphism* between two functors F and G if there is a natural transformation $u : F \to G$ that is an isomorphism in $\textsc{Func}(\mathcal{C}, \mathcal{D})$.

(b) If $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{U}$-categories, $\textsc{Func}(\mathcal{C}, \mathcal{D})$ is not in general a $\mathcal{U}$-category. However, if $\mathcal{C}$ is furthermore $\mathcal{U}$-small, then it can be shown that for any two functors $F, G : \mathcal{C} \to \mathcal{D}$, $\mathrm{Hom}_{\textsc{Func}(\mathcal{C},\mathcal{D})}(F, G)$ injects to a subset of $\prod_{A \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{D}}(FA, GA) \in \mathcal{U}$ with the help of (**I.1.3**(viii) [Mor20]). This implies $\mathrm{Hom}_{\textsc{Func}(\mathcal{C},\mathcal{D})}(F, G) \in \mathcal{U}$, which makes $\textsc{Func}(\mathcal{C}, \mathcal{D})$ a $\mathcal{U}$-category.

1.3.5 Definitions (Types of Functors) (2.42 [Bö20]): Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(a) **Faithful:** F is *faithful* if for all objects A and B in $\mathcal{C}$, the mapping:

$$F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(FA, FB), f \mapsto Ff \text{ is injective.}$$

(b) **Full:** F is *full* if for all objects A and B in $\mathcal{C}$, the mapping:

$$F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(FA, FB), f \mapsto Ff \text{ is surjective.}$$

(c) **Fully Faithful:** F is *fully faithful* if F is full and faithful.

It is easy to check that the image $F(\mathcal{C})$ of a fully faithful functor is a subcategory of $\mathcal{D}$.

(d) **Conservative:** F is *conservative* or *reflects isomorphisms* if for every morphism f in $\mathcal{C}$ whereby Ff is an isomorphism in $\mathcal{D}$, f must then itself be an isomorphism. This definition will be relevant for **Section 3.2**.

(e) **Category-Isomorphism:** F is a *category-isomorphism* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = 1_{\mathcal{C}}$ is the identity functor on $\mathcal{C}$ and analogously $F \circ G = 1_{\mathcal{D}}$.

(f) **Category-Equivalence:** F is a *category-equivalence* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that there exists natural isomorphisms $u = G \circ F \to 1_{\mathcal{C}}$ and $v = F \circ G \to 1_{\mathcal{D}}$.

A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a *category-anti-equivalence from* $\mathcal{C}$ *to* $\mathcal{D}$ if F is a *category-equivalence.*

(g) **Adjoint:** Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{U}$-categories. For functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, we write equivalently that: (i) (F, G) is an *adjunction* or an *adjoint pair*, (ii) F is *left adjoint* to G, (iii) G is *right adjoint* to F, (iv) $F \dashv G$, when there exists a natural isomorphism u between the two functors:

$$\mathrm{Hom}_{\mathcal{D}}(F(\_), \_) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{SET}_{\mathcal{U}}, (C, D) \mapsto \mathrm{Hom}_{\mathcal{D}}(FC, D),$$

$$\mathrm{Hom}_{\mathcal{C}}(\_, G(\_)) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{SET}_{\mathcal{U}}, (C, D) \mapsto \mathrm{Hom}_{\mathcal{C}}(C, GD),$$

This implies that for all objects C in $\mathcal{C}$ and all objects D in $\mathcal{D}$, there is a bijection of sets:

$$u(C, D) : \mathrm{Hom}_{\mathcal{D}}(FC, D) \to \mathrm{Hom}_{\mathcal{C}}(C, GD).$$

1.3.6 Example (Category-Isomorphisms): Since every abelian $\mathcal{U}$-group is canonically a $\mathcal{U}$-right-$\mathbb{Z}$-module and a $\mathcal{U}$-left-$\mathbb{Z}$-module, it is clear that there exists category-isomorphisms $\mathrm{AB}_{\mathcal{U}} \cong {}_{\mathbb{Z}}\mathrm{MOD}_{\mathcal{U}} \cong \mathrm{MOD}_{\mathbb{Z}\mathcal{U}}$.

1.3.7 Examples (Adjunctions):

(a) **Free Functors (7.4** [Bra16]**):** We define the *free functor* $\langle \_ \rangle : \mathrm{SET}_{\mathcal{U}} \to {}_R\mathrm{MOD}_{\mathcal{U}}$ as follows: For a $\mathcal{U}$-set, we define the *free $\mathcal{U}$-left-R-module on A*, denoted by $\langle A \rangle$, as the $\mathcal{U}$-set:

$$\{(x_a)_{a\in A} | \text{ For all } a : x_a \in R, \text{ and for finitely many } a : x_a \neq 0\},$$

equipped with the addition $(x_a)_{a\in A} + (y_a)_{a\in A} = (x_a + y_a)_{a\in A}$ and multiplication $r \cdot (x_a)_{a\in A} = (r \cdot x_a)_{a\in A}$. It is easy to check that $\langle A \rangle$ is an object in ${}_R\mathrm{MOD}_{\mathcal{U}}$. For a function $f : A \to B$ between $\mathcal{U}$-sets, we have the corresponding module homomorphism $\langle f \rangle : \langle A \rangle \to \langle B \rangle$ defined as follows: For $a' \in A$, we define $(\delta_{a,a'})_{a\in A} \in \langle A \rangle$, whereby $\delta_{a,a'} = 1$ if $a = a'$ and $\delta_{a,a'} = 0$ if $a \neq a'$. We then set $\langle f \rangle((\delta_{a,a'})_{a\in A}) = (\delta_{b,f(a')})_{b\in B}$ for all $a' \in A$. This defines $\langle f \rangle$ uniquely as a module homomorphism since $((\delta_{a,a'})_{a\in A})_{a'\in A}$ forms an R-basis of $\langle A \rangle$. It is easy to check that for two functions $f : A \to B$ and $g : B \to C$, we have that $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle$ and thus that $\langle \_ \rangle$ is a well-defined functor.

Let $\mathrm{For} : {}_R\mathrm{MOD}_{\mathcal{U}} \to \mathrm{SET}_{\mathcal{U}}$ be the forgetful functor as defined in (**1.3.2**(b)). We then claim that $(\langle \_ \rangle, \mathrm{For})$ is an adjoint pair, i.e. that there exists a natural isomorphism $u : \mathrm{Hom}_{{}_R\mathrm{MOD}_{\mathcal{U}}}(\langle \_ \rangle, \_) \to \mathrm{Hom}_{\mathrm{SET}_{\mathcal{U}}}(\_, \mathrm{For}(\_))$. Let A be a $\mathcal{U}$-set and M be a $\mathcal{U}$-left-R-module, we then define u through the bijections:

$$u(A, M) : \mathrm{Hom}_{{}_R\mathrm{MOD}_{\mathcal{U}}}(\langle A \rangle, M) \to \mathrm{Hom}_{\mathrm{SET}_{\mathcal{U}}}(A, \mathrm{For}M), f \mapsto f|_A,$$

whereby $f|_A$ is the morphism that maps a' to $f((\delta_{a,a'})_{a\in A})$ for all $a' \in A$. $u(A, M)$ is a bijection with the inverse mapping $v(A, M)$ that sends $g : A \to \mathrm{For}M$ to $g' : \langle A \rangle \to M$ with $g'((x_a)_{a\in A}) = (x_a \cdot g(a))_{a\in A}$. Due to $((\delta_{a,a'})_{a\in A})_{a'\in A}$ being a R-basis of $\langle A \rangle$, it is easy to check that $v(A, M)$ and $u(A, M)$ are inverse to each other. Furthermore, for a morphism of $\mathcal{U}$-sets $f : A \to B$ and a module homomorphism $g : M \to N$ of $\mathcal{U}$-left-R-modules, it is easy to check that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{{}_R\mathrm{MOD}_{\mathcal{U}}}(\langle B \rangle, M) & \xrightarrow{u(B,M)} & \mathrm{Hom}_{\mathrm{SET}_{\mathcal{U}}}(B, \mathrm{For}M) \\ {\scriptstyle (\langle f \rangle^*, g_*)}\downarrow & & \downarrow{\scriptstyle (f^*, (\mathrm{For}(g))_*)} \\ \mathrm{Hom}_{{}_R\mathrm{MOD}_{\mathcal{U}}}(\langle A \rangle, N) & \xrightarrow{u(A,N)} & \mathrm{Hom}_{{}_R\mathrm{MOD}_{\mathcal{U}}}(A, \mathrm{For}N) \end{array}.$$

The above construction functions for $\mathcal{U}$-right-R-modules as well, implying that the analogously defined *free functor* $\langle \_ \rangle : \mathrm{SET}_{\mathcal{U}} \to \mathrm{MOD}_{R\mathcal{U}}$ and forgetful functor $\mathrm{For} : \mathrm{MOD}_{R\mathcal{U}} \to \mathrm{SET}_{\mathcal{U}}$ form an adjoint pair $(\langle \_ \rangle, \mathrm{For})$.

(b) **Hom-Tensor Adjunctions (**[Sta21]**):** Let R be a commutative $\mathcal{U}$-ring and M be an object in ${}_R\mathrm{MOD}_{\mathcal{U}}$, then the functors $((\_ \otimes_R M), \mathrm{Hom}_{{}_R\mathrm{MOD}_{\mathcal{U}}}(M, \_))$ form an adjunction.

Now we state and prove a useful category-isomorphism using opposite algebraic structures.

1.3.8 Definitions (Opposite Algebraic Structures):

(a) **Opposite Groups:** Let $(G, 0, +)$ be a group. The *opposite group* $(G^{\mathrm{op}}, 0, +^{\mathrm{op}})$, written G^{op}, has the same underlying set $G^{\mathrm{op}} = G$. However, we define the addition as $a +^{\mathrm{op}} b = b + a$ for all $a, b \in G$. This defines a unique group G^{op} with the same inverses (i.e. $-^{\mathrm{op}}a = -a$) and the same neutral element 0 as G.

The inverse mapping $G \to G^{\mathrm{op}}$, $g \mapsto -g$ defines a group isomorphism between G and G^{op}.

(b) **Opposite Rings (1.1** [Bö20]**):** Analogously, we define the *opposite ring* $(R^{\mathrm{op}}, 0, 1, +, \cdot^{\mathrm{op}})$ to the ring $(R, 0, 1, +, \cdot)$, whereby the underlying subgroup $(R^{\mathrm{op}}, 0, +)$ is the same as $(R, 0, +)$. For $\cdot^{\mathrm{op}}$ we define $a \cdot^{\mathrm{op}} b = b \cdot a$ for all $a, b \in R$.

1.3.9 Lemma (Category-Isomorphisms of Modules) ([Sta15]**):** Let R be a $\mathcal{U}$-ring, then $\textsc{Mod}_{R\mathcal{U}}$ is category-isomorphic to ${}_{R^{\mathrm{op}}}\textsc{Mod}_{\mathcal{U}}$. Thus, many theorems that we prove for ${}_R\textsc{Mod}_{\mathcal{U}}$ for any $\mathcal{U}$-ring R will also apply to $\textsc{Mod}_{R\mathcal{U}}$ for any $\mathcal{U}$-ring R (as we may simply replace R with R^{op}).

Proof: See reference for a general explanation. □

1.4 Presheaves and the Yoneda Lemma

Let $\mathcal{U}$ be a Grothendieck universe and let $\mathcal{C}$ be a $\mathcal{U}$-category.

1.4.1 Definition (Presheaves) (2.39 [Bö20]**):** We define the *category of presheaves on* $\mathcal{C}$ as the category $\mathrm{PS}\textsc{h}(\mathcal{C}) = \textsc{Func}(\mathcal{C}^{\mathrm{op}}, \textsc{Set}_{\mathcal{U}})$. For a $\mathcal{U}$-ring R, we also write $\mathrm{PS}\textsc{h}(\mathcal{C}, R) = \textsc{Func}(\mathcal{C}^{\mathrm{op}}, {}_R\textsc{Mod}_{\mathcal{U}})$. More generally let $\mathcal{D}$ be a $\mathcal{U}$-category, then the *category of $\mathcal{D}$-valued presheaves on* $\mathcal{C}$ is the category $\mathrm{PS}\textsc{h}(\mathcal{C}, \mathcal{D}) = \textsc{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$.

1.4.2 Definition (Categories of Open Sets) (I.3.1.2 [Mor20]**):** Let (X, τ) be a topology, then we define the *category of open sets* $\textsc{Open}(X)$ as the category with the following objects and morphisms:

(i) Open sets $U \in \tau$ are the objects of $\textsc{Open}(X)$.

(ii) Inclusions of the form $\iota : U \to U', u \mapsto u$ for all $U \subset U'$ are the morphisms of $\textsc{Open}(X)$.

If X is a $\mathcal{U}$-topology, then $\textsc{Open}(X)$ is clearly a $\mathcal{U}$-small category.

1.4.3 Examples (Presheaves):

(a) Let A be an object in $\mathcal{C}$. then the hom-functor $\mathrm{Hom}_{\mathcal{C}}(\_, A) : \mathcal{C}^{\mathrm{op}} \to \textsc{Set}_{\mathcal{U}}$, as defined in (**1.3.2**(c)), is a presheaf on $\mathcal{C}$.

(b) Let (X, τ) and (Y, η) be $\mathcal{U}$-topologies. We write $C(X, Y)$ as the *space of continuous functions from X to Y*, which is clearly a $\mathcal{U}$-set. For an open subset $U \subset X$ equipped with the subset topology $(U, \tau|_U)$, $C(U, Y)$ is the space of continuous functions from U to Y.

We can then define the functor $C : \textsc{Open}(X)^{\mathrm{op}} \to \textsc{Set}_{\mathcal{U}}$ whereby $CU = C(U, Y)$. For an inclusion morphism $\iota : U \to U'$, i.e. $U \subset U'$, we define $C\iota : C(U', Y) \to C(U, Y)$ as the mapping $f \mapsto f|_U$ that restricts functions onto a smaller domain. It is easy to show that C is a well-defined presheaf. C is also a motivating example for sheaves, which we will encounter later in **Section 4**.

(c) Examples of presheaves also exist in algebraic geometry. Let k be an algebraically closed $\mathcal{U}$-field and let X be a $\mathcal{U}$-variety over k equipped with the Zariski topology. We define the *structure presheaf* as $\Gamma_X : \textsc{Open}(X)^{\mathrm{op}} \to \textsc{Set}_{\mathcal{U}}$, whereby for an inclusion $\iota : U \to U'$ of morphisms we have:

$$\Gamma_X(U) = \{f : U \to k | f \text{ is a regular function}\}, \quad \Gamma_X\iota : \Gamma_X(U') \to \Gamma_X(U), f \mapsto f|_U.$$

1.4.4 Definition (Representable) (5.2.12 [Bra16]**):** A presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{SET}_{\mathcal{U}}$ that is naturally isomorphic to a presheaf of the form $h_A = \mathrm{Hom}_{\mathcal{C}}(\_, A)$ for an object A in $\mathcal{C}$, as defined in (**1.3.2**(c)), is called *representable.*

1.4.5 Theorem (Yoneda Lemma) (5.2.5 [Bra16]**), (A.1.5** [Mor20]**), (3.1 Theorem 2** [RCT13]**):** Let A be an object in $\mathcal{C}$ and $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{SET}_{\mathcal{U}}$ be a presheaf on $\mathcal{C}$, then the following bijection between $\mathcal{U}$-sets exists:

$$\Phi : \mathrm{Hom}_{\mathrm{PSH}(\mathcal{C})}(h_A, F) \to FA, (u : h_A \to F) \mapsto uA(1_A) \in FA.$$

Proof: See references. □

1.4.6 Definition (Yoneda Embeddings) (5.2.10 [Bra16]**), (3.1.1** [RCT13]**):** The *Yoneda embedding* is a functor $h : \mathcal{C} \to \mathrm{PSH}(\mathcal{C})$ such that:

(i) h sends an object W to the hom-functor h_W, i.e. $hW = h_W$

(ii) Let $f : A \to B$ be a morphism in $\mathcal{C}$, we then define $hf = h_f : h_A \to h_B$, i.e. $h_f : \mathrm{Hom}_{\mathcal{C}}(\_, A) \to \mathrm{Hom}_{\mathcal{C}}(\_, B)$, as the natural transformation defined through the left composition morphisms $(f_* = h_f W : \mathrm{Hom}_{\mathcal{C}}(W, A) \to \mathrm{Hom}_{\mathcal{C}}(W, B))_{W \in \mathrm{Ob}(\mathcal{C})}$. h_f is well-defined as a natural transformation as it is clear for all morphisms $k : W \to V$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} h_A V & \xrightarrow{h_f V = f_*} & h_B V \\ {\scriptstyle h_A k = k^*}\downarrow & & \downarrow{\scriptstyle h_B k = k^*} \\ h_A W & \xrightarrow{h_f W = f_*} & h_B W \end{array} .$$

It is also easy to check that for another morphism $g : B \to C$ in $\mathcal{C}$, we have $h_{g \circ f} = h_g \circ h_f$ due to the functoriality of left composition.

1.4.7 Corollary (Yoneda Lemma): The functor $h : \mathcal{C} \to \mathrm{PSH}(\mathcal{C})$ is fully faithful.

Proof: For all objects A and B in $\mathcal{C}$, we want to show that $h_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathrm{PSH}(\mathcal{C})}(h_A, h_B), f \mapsto h_f$ is bijective. Due to the *Yoneda lemma* from (**1.4.5**) with $h_B = F$, we have the bijection $\Phi : \mathrm{Hom}_{\mathrm{PSH}(\mathcal{C})}(h_A, h_B) \to h_B A = \mathrm{Hom}_{\mathcal{C}}(A, B)$. With $\Phi(h_f) = h_f A(1_A) = f$, we see that $h_{A,B} = \Phi^{-1} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathrm{PSH}(\mathcal{C})}(h_A, h_B)$ is bijective. □

1.4.8 Definition (Representations) (5.2.13 [Bra16]**):** A representable presheaf F of $\mathcal{C}$ has the *representation* (A, α), whereby A is an object of $\mathcal{C}$ and $\alpha \in FA$ such (A, α) describes a natural isomorphism $\Phi^{-1}(\alpha) : h_A \to F$.

1.4.9 Note (Dual Yoneda Lemma) (5.2.9 [Bra16]**), (3.1 Theorem 1** [RCT13]**):** One can replace $\mathcal{C}^{\mathrm{op}}$ with $\mathcal{C}$ and do the same calculations from **Section 1.4** again. In this case we would use $k_A = \mathrm{Hom}_{\mathcal{C}}(A, \_) : \mathcal{C} \to \mathrm{SET}_{\mathcal{U}}$ instead of h_A and say that a natural transformation in $\mathrm{FUNC}(\mathcal{C}, \mathrm{SET}_{\mathcal{U}})$ is *corepresentable* if it is naturally isomorphic to k_A for an object A in $\mathcal{C}$.

Analogously, there exists a bijection due to the *dual Yoneda lemma*:

$$\Psi : \mathrm{Hom}_{\mathrm{FUNC}(\mathcal{C}, \mathrm{SET}_{\mathcal{U}})}(k_A, F) \to FA, (u : k_A \to F) \mapsto uA(1_A),$$

such that corepresentable functors $F : \mathcal{C} \to \mathrm{SET}_{\mathcal{U}}$ can be *corepresented* with the pair (A, α), whereby $\Psi^{-1}(\alpha) : k_A \to F$ is an isomorphism.

Just as in (**1.4.7**), it can be shown that the induced functor $k : \mathcal{C}^{\mathrm{op}} \to \mathrm{FUNC}(\mathcal{C}, \mathrm{SET}_{\mathcal{U}})$, which sends an object W to the hom-functor k_W, is fully faithful due to the *dual Yoneda lemma.*

1.5 Limits and Colimits

Limits and colimits are important constructions that allow us to understand universal properties found in mathematical constructions in a diagrammatic way. Let $\mathcal{U}$ be a Grothendieck universe and $\mathcal{C}$ be a $\mathcal{U}$-category.

1.5.1 Definition (Diagrams): We define a functor $D : \mathcal{I} \to \mathcal{C}$ as a *$\mathcal{U}$-small diagram* when $\mathcal{I}$ is a $\mathcal{U}$-small category.

$\mathcal{I}$ as an index category is sometimes only defined up to a category-isomorphism. $\mathcal{I}$ also sometimes has a finite set of objects, in which case we call $\mathcal{I}$ *finite* and D a *finite $\mathcal{U}$-small diagram.*

1.5.2 Definitions (Cones and Cocones) (6.2.1, 6.3.1 [Bra16]**),** ([Kie06])**:**

(a) **Cones:** Let $D : \mathcal{I} \to \mathcal{C}$ be a $\mathcal{U}$-small diagram and let T be an object in $\mathcal{C}$. A *D-cone from T to $\mathcal{C}$* is a collection of morphisms $(p_i : T \to \mathcal{C}i)_{i \in \mathrm{Ob}(\mathcal{I})}$ indexed by $\mathrm{Ob}(\mathcal{I})$, such that for every morphism $f : i \to j$ in $\mathcal{I}$, the following diagram commutes:

$$\begin{array}{ccc} & T & \\ {\scriptstyle p_i}\swarrow & & \searrow{\scriptstyle p_j} \\ \mathcal{C}i & \xrightarrow[Df]{} & \mathcal{C}j \end{array} .$$

(b) **Cocones:** Analogously we define cocones. Let $D : \mathcal{I} \to \mathcal{C}$ be a $\mathcal{U}$-small diagram and let T be an object in $\mathcal{C}$. A *D-cocone from $\mathcal{C}$ to T* is a collection of morphisms $(p_i : \mathcal{C}i \to T)_{i \in \mathrm{Ob}(\mathcal{I})}$ indexed by $\mathrm{Ob}(\mathcal{I})$, such that for every morphism $f : i \to j$ in $\mathcal{I}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}i & \xrightarrow{Df} & \mathcal{C}j \\ {\scriptstyle p_i}\searrow & & \swarrow{\scriptstyle p_j} \\ & T & \end{array} .$$

1.5.3 Definition (Constant Diagrams) (6.2.1 [Bra16]**):** Let $\mathcal{C}$ be a $\mathcal{U}$-category and T be an object in $\mathcal{C}$. For any $\mathcal{U}$-small index category $\mathcal{I}$ we define $\triangle(T) : \mathcal{I} \to \mathcal{C}$ as the *constant diagram* that sends all objects in $\mathcal{I}$ to T and all morphisms in $\mathcal{I}$ to the identity 1_T. The family of identities $(1_T : T \to T)_{i \in \mathrm{Ob}(\mathcal{I})}$ is a $\triangle(T)$-*constant cone* and a $\triangle(T)$-*constant cocone.*

Furthermore, for a morphism $f : T \to T'$ in $\mathcal{C}$, there exists a canonical natural transformation $\triangle(f) : \triangle(T) \to \triangle(T')$ given by the morphisms $(f : T \to T')_{i \in \mathrm{Ob}(\mathcal{I})}$.

1.5.4 Note (Constant Diagrams): Cones and cocones can be described with the help of constant diagrams. Let $D : \mathcal{I} \to \mathcal{C}$ be a $\mathcal{U}$-small diagram. A D-cone from T to $\mathcal{C}$ can be uniquely identified as a natural transformation from $\triangle(T)$ to D. A D-cocone from $\mathcal{C}$ to T is uniquely identified as a natural transformation from D to $\triangle(T)$. These cones and cocones lie in $\mathrm{Hom}_{\textsc{Func}(\mathcal{I},\mathcal{C})}(\triangle(T), D)$ and $\mathrm{Hom}_{\textsc{Func}(\mathcal{I},\mathcal{C})}(D, \triangle(T))$ respectively.

1.5.5 Definitions (Limits and Colimits) (6.2.2, 6.2.3, 6.3.2, 6.3.3 [Bra16]**),** ([Kie06])**:**

(a) **Limits:** A *limit of a $\mathcal{U}$-small diagram $D : \mathcal{I} \to \mathcal{C}$* is a representation of the functor $\mathrm{Hom}_{\textsc{Func}(\mathcal{I},\mathcal{C})}(\triangle(\_), D) : \mathcal{C}^{\mathrm{op}} \to \textsc{Set}_{\mathcal{U}}$. Explicitly, the representation is (T, α) whereby T is an object in $\mathcal{C}$ and α is an element of $\textsc{Func}(\triangle(T), D)$, i.e. α is a D-cone from T to $\mathcal{C}$.

To demystify this formal definition, we observe that we have found a natural isomorphism between functors $\mathrm{Hom}_{\textsc{Func}(\mathcal{I},\mathcal{C})}(\triangle(\_), D)$ and $\mathrm{Hom}_{\mathcal{C}}(\_, T) \cong \mathrm{Hom}_{\textsc{Func}(\mathcal{I},\mathcal{C})}(\triangle(\_), \triangle(T))$ as seen in (**1.4.4**). The proof of the *Yoneda lemma* (**1.4.5**) gives us the insight that the pair (T, α) is a limit if and only if for every D-cone $\triangle(A) \to D$ from A to $\mathcal{C}$, there exists exactly one morphism $\beta : A \to T$ in $\mathcal{C}$ such that the composition $\alpha \circ \triangle(\beta) : \triangle(A) \to D$ is the same as the original D-cone $\triangle(A) \to D$.

We denote the limit as $(T, \alpha) = \lim_{\mathcal{I}} D$, we often imprecisely denote $T = \lim_{\mathcal{I}} D$ and leave α to be found contextually.

(b) **Colimits:** A *colimit of a $\mathcal{U}$-small diagram* $D : \mathcal{I} \to \mathcal{C}$ is a corepresentation of the functor $\mathrm{Hom}_{\mathrm{FUNC}(\mathcal{I},\mathcal{C})}(D, \triangle(\_)) : \mathcal{C} \to \mathrm{SET}_{\mathcal{U}}$. Explicitly, the corepresentation is (T, α) whereby T is an object in $\mathcal{C}$ and α is a D-cocone from $\mathcal{C}$ to T.

As with limits, we observe that we have found a natural isomorphism between functors $\mathrm{Hom}_{\mathrm{FUNC}(\mathcal{I},\mathcal{C})}(D, \triangle(\_))$ and $\mathrm{Hom}_{\mathrm{FUNC}(\mathcal{I},\mathcal{C})}(\triangle(T), \triangle(\_)) \cong \mathrm{Hom}_{\mathcal{C}}(T, \_)$ as seen in (**1.4.9**). The proof of the *dual Yoneda lemma* and (**1.4.9**) gives us the insight that the pair (T, α) is a colimit if and only if for every D-cocone $D \to \triangle(A)$ from $\mathcal{C}$ to A, there exists exactly one morphism $\beta : T \to A$ in $\mathcal{C}$ such that the composition $\triangle(\beta) \circ \alpha : D \to \triangle(A)$, is the same as our original D-cocone $D \to \triangle(A)$.

We denote the colimit as $(T, \alpha) = \mathrm{colim}_{\mathcal{I}} D$, we often imprecisely denote $T = \mathrm{colim}_{\mathcal{I}} D$ and leave α to be found contextually.

1.5.6 Note (Existence of Limits and Colimits): The limit (or colimit) (T, α) of a diagram D does not always exist and if it does, it is not always uniquely defined. However, one can show that they are unique up to an isomorphism in $\mathcal{C}$ i.e. if (T', α') is another limit (or colimit), then one can find an isomorphism $f : T \to T'$ in $\mathcal{C}$ such that $\alpha : \triangle(T) \to D$ and $\alpha' \circ \triangle(f) : \triangle(T) \to D$ (or $\triangle(f) \circ \alpha : D \to \triangle(T')$ and $\alpha' : D \to \triangle(T')$ for colimits) are isomorphic to each other as natural transformations.

This is important to keep in mind when we define objects with universal properties such as kernels, equalizers and fiber products using limits (or colimits), as they are unique only up to isomorphism.

1.5.7 Lemma (Limits and Colimits as Functors) (I.5.1.4, I.5.1.5 [Mor20]**):** For a fixed $\mathcal{U}$-small index category $\mathcal{I}$ and a $\mathcal{U}$-category $\mathcal{C}$ that contains all limits of the diagram $\mathcal{I}$ (respectively all colimits of the diagram $\mathcal{I}$), limits (respectively colimits) define functors uniquely up to an isomorphism:

$$\lim_{\mathcal{I}} : \mathrm{FUNC}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}, D \mapsto \lim_{\mathcal{I}} D, \quad \mathrm{colim}_{\mathcal{I}} : \mathrm{FUNC}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}, D \mapsto \mathrm{colim}_{\mathcal{I}} D.$$

Proof: The object mappings $D \mapsto \lim_{\mathcal{I}} D$ (respectively $D \mapsto \mathrm{colim}_{\mathcal{I}} D$) are defined up to isomorphism as $\mathcal{C}$ contains all limits of the diagram $\mathcal{I}$ (respectively all colimits of the diagram $\mathcal{I}$).

<u>For limits:</u> For any natural transformation $u : D \to E$ of functors $D, E : \mathcal{I} \to \mathcal{C}$, we need a unique morphism from D-cones $\lim_{\mathcal{I}} D = (\lim_{\mathcal{I}} D, \alpha)$ to $\lim_{\mathcal{I}} E = (\lim_{\mathcal{I}} E, \beta)$.

For the D-cone $\alpha \in \mathrm{FUNC}(\triangle(\lim_{\mathcal{I}} D), D)$, we see that $u \circ \alpha \in \mathrm{FUNC}(\triangle(\lim_{\mathcal{I}} D), E)$ is a D-cone from $\lim_{\mathcal{I}} D$ to $\mathcal{C}$. Due to the universal properties of a limit, there exists a unique morphism $\gamma : \lim_{\mathcal{I}} D \to \lim_{\mathcal{I}} E$ in $\mathcal{C}$ such that $u \circ \alpha = \beta \circ \triangle(\gamma)$ for the limit cone $\beta \in \mathrm{FUNC}(\triangle(\lim_{\mathcal{I}} E), E)$.

γ is our definition for $\lim_{\mathcal{I}} u : \lim_{\mathcal{I}} D \to \lim_{\mathcal{I}} E$. The uniqueness of γ to the objects $\lim_{\mathcal{I}} D$ and $\lim_{\mathcal{I}} E$ guarantees that for another natural transformation $v : E \to F$, we have $\lim_{\mathcal{I}} v \circ \lim_{\mathcal{I}} u = \lim_{\mathcal{I}}(v \circ u)$. $\lim_{\mathcal{I}}$ is thus a functor.

<u>For colimits:</u> The case for colimits follows analogously. □

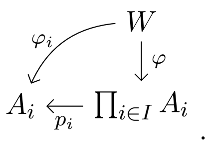
1.5.8 Definitions (Limits and Colimits of Diagrams) (2.5.1, 6.2.4, 6.3.5 [Bra16]**):** Let $\mathcal{C}$ be a $\mathcal{U}$-category.

(a) **Initial Objects:** An *initial object* A is an object in $\mathcal{C}$ such that for every object W in $\mathcal{C}$, there exists exactly one morphism in $\mathcal{C}$ from A to W ($A \to W$). This is equivalent to saying that A is the limit of the empty diagram $D : \mathcal{I} = \emptyset \to \mathcal{C}$.

(b) **Terminal Objects:** A *final* or *terminal object* A is an object in $\mathcal{C}$ such that for every object W in $\mathcal{C}$, there exists exactly one morphism in $\mathcal{C}$ from W to A ($W \to A$). This is equivalent to saying that A is the colimit of the empty diagram $D : \mathcal{I} = \emptyset \to \mathcal{C}$.

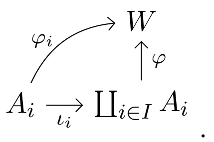
(c) **Zero Objects:** An object A in $\mathcal{C}$ that is both initial and terminal is called a *zero object*. We denote zero objects with 0.

(d) **Products:** For objects A_i in $\mathcal{C}$ indexed by a $\mathcal{U}$-set I, a *(categorical $\mathcal{U}$-small) product* of $(A_i)_{i\in I}$ consists of an object $\prod_{i\in I} A_i$ in $\mathcal{C}$ with projection morphisms $(p_i : \prod_{i\in I} A_i \to A_i)_{i\in I}$, such that for all objects W in $\mathcal{C}$ and all morphisms $(\varphi_i : W \to A_i)_{i\in I}$ in $\mathcal{C}$, there exists exactly one morphism φ in $\mathcal{C}$ such that the following diagram commutes for all $i \in I$:



More formally, $(\prod_{i\in I} A_i, (p_i)_{i\in I})$ is the limit of the diagram $D : \mathcal{I} \to \mathcal{C}, i \mapsto A_i$ whereby $\mathrm{Ob}(\mathcal{I}) = I$ and $\mathrm{Mor}(\mathcal{I})$ consists only of identities.

(e) **Coproducts:** Analogously we may define the coproduct as follows: For objects A_i in $\mathcal{C}$ indexed by a $\mathcal{U}$-set I, a *(categorical $\mathcal{U}$-small) coproduct* of $(A_i)_{i\in I}$ consists of an object $\coprod_{i\in I} A_i$ in $\mathcal{C}$ with inclusion morphisms $(\iota_i : A_i \to \coprod_{i\in I} A_i)_{i\in I}$, such that for all objects W in $\mathcal{C}$ and all morphisms $(\varphi_i : A_i \to W)_{i\in I}$ in $\mathcal{C}$, there exists exactly one morphism φ in $\mathcal{C}$ such that the following diagram commutes for all $i \in I$:



$(\coprod_{i\in I} A_i, (\iota_i)_{i\in I})$ is the colimit of the diagram $D : \mathcal{I} \to \mathcal{C}, i \mapsto A_i$ whereby $\mathrm{Ob}(\mathcal{I}) = I$ and $\mathrm{Mor}(\mathcal{I})$ consists only of identities.

1.5.9 Examples (Limits and Colimits of Diagrams):

(a) One can easily show that $\emptyset$ is an initial object in $\mathrm{SET}_{\mathcal{U}}$ and that any set containing only one object, i.e. a *singleton* $\{\star\}$, is a terminal object in $\mathrm{SET}_{\mathcal{U}}$. In $\mathrm{AB}_{\mathcal{U}}$ it can be shown that the trivial group $\{e\}$ is both initial and terminal, i.e. a zero object.

(b) Let R be a ring, then ${}_R\mathrm{MOD}_{\mathcal{U}}$ (and $\mathrm{MOD}_{R\mathcal{U}}$) have all $\mathcal{U}$-small products and coproducts (i.e. such products and coproducts exist and are well-defined). Let I be a $\mathcal{U}$-small set and $(M_i)_{i\in I}$ be a collection of modules in ${}_R\mathrm{MOD}_{\mathcal{U}}$ (or $\mathrm{MOD}_{R\mathcal{U}}$). For products, $\prod_{i\in I} M_i = \{(m_i)_{i\in I} | m_i \in M_i\}$ with the canonical projections defines a categorical product of $(M_i)_{i\in I}$ in ${}_R\mathrm{MOD}_{\mathcal{U}}$ (or $\mathrm{MOD}_{R\mathcal{U}}$). Analogously for coproducts, $\bigoplus_{i\in I} M_i = \{(m_i)_{i\in I} | m_i \in M_i,\ m_i \neq 0$ for finitely many $i\}$ with the canonical inclusions defines a categorical coproduct of $(M_i)_{i\in I}$ in ${}_R\mathrm{MOD}_{\mathcal{U}}$ (or $\mathrm{MOD}_{R\mathcal{U}}$).

1.5.10 Note (Hom-Functors Commute with Limits and Colimits) (I.5.3.3, I.5.3.5 [Mor20]**):** Let $\mathcal{I}$ be a $\mathcal{U}$-small index category such that for all diagrams $D : \mathcal{I} \to \mathcal{C}$, we have that the limit $\lim_{\mathcal{I}} D$ exists in $\mathcal{C}$ (we denote this as a property of $\mathcal{C}$, that $\mathcal{C}$ *contains all $\mathcal{I}$-indexed limits*). Let A be an object in $\mathcal{C}$, then for the functor $\mathrm{Hom}_{\mathcal{C}}(A, \_)$, it is shown in (**I.5.3.3** [Mor20]) that for all diagrams $D : \mathcal{I} \to \mathcal{C}$, we have an isomorphism in $\mathrm{SET}_{\mathcal{U}}$:

$$\mathrm{Hom}_{\mathcal{C}}(A, \lim_{\mathcal{I}} D) \cong \lim_{i\in \mathrm{Ob}(\mathcal{I})}(\mathrm{Hom}_{\mathcal{C}}(A, D(i))).$$

This can be denoted as follows: $\mathrm{Hom}_{\mathcal{C}}(A, \_)$ *commutes with $\mathcal{I}$-indexed limits.*

Analogous to (**I.5.3.3** [Mor20]), we have the dual version of this statement as seen in (**I.5.3.5** [Mor20]): For a $\mathcal{U}$-category $\mathcal{C}$ with all $\mathcal{I}$-indexed colimits, for all diagrams $D : \mathcal{I} \to \mathcal{C}$ and an object A in $\mathcal{C}$, we have that:

$$\mathrm{Hom}_{\mathcal{C}}(\lim_{\mathcal{I}^{\mathrm{op}}} D^{\mathrm{op}}, A) = \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{I}} D, A) \cong \lim_{i\in \mathrm{Ob}(\mathcal{I}^{\mathrm{op}})}(\mathrm{Hom}_{\mathcal{C}}(D^{\mathrm{op}}(i), A)),$$

with D^{op} being the opposite functor of D as defined in (**1.3.2**(a)). This can be denoted as follows: $\text{Hom}_{\mathcal{C}}(\_, A) : \mathcal{C}^{\text{op}} \to \text{SET}_{\mathcal{U}}$ *commutes with* $\mathcal{I}^{op}$*-indexed limits.*

1.5.11 Lemma (Explicit Constructions of Limits and Colimits) (I.5.2.1 [Mor20]**):** $\text{SET}_{\mathcal{U}}$ has all $\mathcal{U}$-small indexed limits and $\mathcal{U}$-small indexed colimits (i.e. such limits and colimits exist and are well-defined). More explicitly, let $\mathcal{I}$ be a $\mathcal{U}$-small index category with $\text{Ob}(\mathcal{I}) = I$ and let $D : \mathcal{I} \to \text{SET}_{\mathcal{U}}$ be a diagram. Then the following statements apply:

(a) We claim that the categorical product $\prod_{i\in I} Di$ coincides with the cartesian product, more precisely:

$$A = \left\{ (d_i)_{i\in I} \in \prod_{i\in I} Di | \text{ For all } f : i \to j \text{ in } \mathcal{I} : Df(d_i) = d_j \right\},$$

with the corresponding projections to the i-th component $(g_i : A \to Di)_{i\in I}$, is the limit $\lim_{\mathcal{I}} D$.

(b) The categorial coproduct $\coprod_{i\in I} Di$, which is the same as the disjoint union, exists in $\text{SET}_{\mathcal{U}}$. We define an equivalence relation $\sim$ on $\coprod_{i\in I} Di$ as the equivalence relation *generated* from the following relations:

For $a \in Di$ and $b \in Dj$ with $f : i \to j$ in $\mathcal{I}$ such that $Df(a) = b \in Dj$, then $a \sim b$.

Since the above relation only explicitly fulfills reflexivity, we add all the minimum extra relations into $\sim$, such that $\sim$ also fulfills symmetry and transitivity, and is thus an equivalence relation (see (**I.5.2.1** [Mor20]) for an explicit description of $\sim$). We claim that $A = \coprod_{i\in I} Di/\sim$ is the colimit $\text{colim}_{\mathcal{I}} D$ with the canonical inclusions $(g_i : Di \to A)_{i\in I}$.

Proof: For (a): For all D-cones from T to $\text{SET}_{\mathcal{U}}$ with functions $(p_i : T \to Di)_{i\in I}$, we have $Df \circ p_i = p_j$ for all morphisms $f : i \to j$ in $\mathcal{I}$. There exists exactly one function $\beta : T \to A$ given as $\beta = (p_i)_{i\in I}|^A$ such that the D-cone $(p_i : T \to M_i)_{i\in I}$ is the same as the D-cone $(g_i \circ \beta : T \to Di)_{i\in I}$.

For (b): For all D-cocones from $\text{SET}_{\mathcal{U}}$ to T with functions $(p_i : Di \to T)_{i\in I}$, we have $p_i = p_j \circ Df$ for all morphisms $f : i \to j$ in $\mathcal{I}$. Due to the construction of $\sim$, the functions $(p_i : Di \to T)_{i\in I}$ induce a function $\beta : A \to T$ as follows, $\beta : [a] \mapsto p_i(a)$ for $a \in Di$ and $[a] \in A$. This mapping is independent of the choice of $a \in [a]$ and is uniquely defined such that the D-cocone $(p_i : Di \to T)_{i\in I}$ is the same as the D-cocone $(\beta \circ g_i : Di \to T)_{i\in I}$. ☐

1.5.12 Lemma (Explicit Constructions of Limits and Colimits) (I.5.5.1 [Mor20]**):** Let R be a $\mathcal{U}$-ring. ${}_R\text{MOD}_{\mathcal{U}}$ and $\text{MOD}_{R\mathcal{U}}$ have all $\mathcal{U}$-small indexed limits and $\mathcal{U}$-small indexed colimits (i.e. such limits and colimits exist and are well-defined). More explicitly, let $\mathcal{I}$ be a $\mathcal{U}$-small index category with $\text{Ob}(\mathcal{I}) = I$ and $M : \mathcal{I} \to {}_R\text{MOD}_{\mathcal{U}}$ (respectively $M : \mathcal{I} \to \text{MOD}_{R\mathcal{U}}$) be a diagram. We will write $Mi = M_i$ for objects $i \in I$. The following statements apply:

(a) The product $\prod_{i\in I} M_i$ in ${}_R\text{MOD}_{\mathcal{U}}$ (or $\text{MOD}_{R\mathcal{U}}$) exists due to (**1.5.9**(b)). Then we claim that:

$$A = \left\{ (m_i)_{i\in I} \in \prod_{i\in I} M_i | \text{ For all } f : i \to j \text{ in } \mathcal{I} : Mf(m_i) = m_j \right\},$$

with the corresponding projections to the i-th component $(g_i : A \to M_i)_{i\in I}$, is the limit $\lim_{\mathcal{I}} M$.

(b) Coproducts exist in ${}_R\text{MOD}_{\mathcal{U}}$ (or $\text{MOD}_{R\mathcal{U}}$) due to (**1.5.9**(b)). Define the coproducts with inclusions $B = (\bigoplus_{i\in I} M_i, (\iota_i)_{i\in I})$ and $B' = (\bigoplus_{f\in\text{Mor}(\mathcal{I})} M_{\text{dom}(f)}, (\iota'_f)_{f\in\text{Mor}(\mathcal{I})})$. We then define the R-linear mappings $\alpha, \beta : B' \to B$ induced by the components:

(i) For all morphisms $f : i \to j$ in $\mathcal{I}$ and for all $x \in M_{\text{dom}(f)} \subset B'$ with $M_{\text{dom}(f)}$ on the f-component of B' : $\alpha(x) = x \in M_i \subset B$.

(ii) For all morphisms $f : i \to j$ in $\mathcal{I}$ and for all $x \in M_{\mathrm{dom}(f)} \subset B'$ with $M_{\mathrm{dom}(f)}$ on the f-component of $B' : \beta(x) = Mf(x) \in M_j \subset B$.

We define $A = B/\mathrm{Im}(\alpha - \beta)$ and the quotient mapping $g : B \to A$, with the mappings $(p'_i : M_i \to A)_{i\in I}$ whereby $p'_i = g \circ \iota_i : M_i \to A$. A with the morphisms $(p'_i)_{i\in I}$ is the colimit $\mathrm{colim}_{\mathcal{I}} M$.

These constructions also directly work for $\text{Ab}_{\mathcal{U}}$, as abelian $\mathcal{U}$-groups are trivially $\mathcal{U}$-left-$\mathbb{Z}$-modules and also $\mathcal{U}$-right-$\mathbb{Z}$-modules.

Proof: Due to (**1.3.9**) and since category-isomorphisms preserve limits and colimits, it is enough to show that ${}_R\text{Mod}_{\mathcal{U}}$ has all limits and colimits.

For (a): For all M-cones from T to ${}_R\text{Mod}_{\mathcal{U}}$ with morphisms $(p_i : T \to M_i)_{i\in I}$, we have $Mf \circ p_i = p_j$ for all morphisms $f : i \to j$ in $\mathcal{I}$. There exists exactly one R-linear mapping $\beta : T \to A$ given as $\beta = (p_i)_{i\in I}|^A$ such that the M-cone $(p_i : T \to M_i)_{i\in I}$ is the same as the M-cone $(g_i \circ \beta : T \to M_i)_{i\in I}$.

For (b): For all M-cocones from ${}_R\text{Mod}_{\mathcal{U}}$ to T with morphisms $(p_i : M_i \to T)_{i\in I}$, we have $p_i = p_j \circ Mf$ for all morphisms $f : i \to j$ in $\mathcal{I}$. Let $p : B \to T$ be the morphism induced from the morphisms $(p_i)_{i\in I}$. For all morphisms f in $\mathcal{I}$ and for all $x \in M_{\mathrm{dom}(f)} \subset B'$ with $M_{\mathrm{dom}(f)}$ on the f-component of B', we have that $p\circ(\alpha-\beta)(x) = p(x - Mf(x)) = p_{\mathrm{dom}(f)}(x) - p_{\mathrm{cod}(f)}(Mf(x)) = 0$. Thus, the image of $\alpha - \beta : B' \to B$ in B is a submodule of the kernel of $p : B \to T$.

Then due to the fundamental theorem of homomorphisms, p factorizes uniquely through $A = B/\mathrm{Im}(\alpha - \beta)$, i.e. there exists a unique R-linear mapping $p' : A \to T$ such that $p = p' \circ g$. Then for the M-cocone $(p_i : M_i \to T)_{i\in I}$ we have $(p' \circ p'_i : M_i \to T)_{i\in I} = (p' \circ g \circ \iota_i : M_i \to T)_{i\in I} = (p \circ \iota_i : M_i \to T)_{i\in I} = (p_i : M_i \to T)_{i\in I}$. Thus A with the morphisms $(p'_i)_{i\in I}$ is the colimit $\mathrm{colim}_{\mathcal{I}} M$. □

1.5.13 Note (Explicit Constructions of Limits and Colimits) (I.5.5.1 [Mor20]**):** Similar proofs show that $\text{Top}_{\mathcal{U}}$ and the *category of $\mathcal{U}$-rings* $\text{Rng}_{\mathcal{U}}$ also have all $\mathcal{U}$-small indexed limits and $\mathcal{U}$-small indexed colimits with explicit constructions.

1.5.14 Definitions (Filtered and Cofiltered Categories) (I.5.6.1 [Mor20]**):** A $\mathcal{U}$-small index category $\mathcal{I}$ is *filtered* if:

(i) $\mathcal{I}$ is not empty, i.e. there exists an object in $\mathcal{I}$.

(ii) For two objects i, j in $\mathcal{I}$, there exists an object k in $\mathcal{I}$ and morphisms $f : i \to k$ and $g : j \to k$ in $\mathcal{I}$.

(iii) If $f, g : i \to j$ are morphisms in $\mathcal{I}$, there exists a morphism $h : j \to k$ so that $h \circ f = h \circ g$.

The dual version also exists, whereby $\mathcal{I}^{\mathrm{op}}$ would be called *cofiltered*.

1.5.15 Definitions (Filtered and Cofiltered Limits and Colimits):

(a) A limit $\lim_{\mathcal{I}} F$ is called *filtered* (or *cofiltered*) if $\mathcal{I}$ is filtered (or cofiltered).

(b) A colimit $\mathrm{colim}_{\mathcal{I}} F$ is called *filtered* (or *cofiltered*) if $\mathcal{I}$ is filtered (or cofiltered).

1.5.16 Lemma (Filtered Colimits for Sets) (I.5.6.2 [Mor20]**):** Let $\mathcal{I}$ be a $\mathcal{U}$-small filtered category and let $D : \mathcal{I} \to \text{Set}_{\mathcal{U}}$ be a diagram. The colimit $(\mathrm{colim}_{\mathcal{I}} D, (g_i : Di \to \mathrm{colim}_{\mathcal{I}} D)_{i\in\mathrm{Ob}(\mathcal{I})})$, which exists in $\text{Set}_{\mathcal{U}}$ due to (**1.5.11**(b)), has the following explicit construction $\mathrm{colim}_{\mathcal{I}} D = (\coprod_{i\in\mathrm{Ob}(\mathcal{I})} Di)/\sim'$. With the equivalence relation $a \in Di \sim' b \in Dj$ if and only if there exists morphisms $f : i \to k$ and $g : j \to k$ in $\mathcal{I}$ such that $Df(a) = Dg(b) \in Dk$. The corresponding morphisms $g_i : Di \to \mathrm{colim}_{\mathcal{I}} D$ must be the canonical inclusions in this construction.

Proof: For the statement to be true, we have to check that $\sim'$ is an equivalence relation, and that $\sim'$ induces the same equivalence classes as those of $\sim$ seen in (**1.5.11**(b)).

For $\sim'$ being an equivalence class: $\sim'$ is clearly reflexive and symmetric so we need to show that $\sim'$ is transitive. Let $a \in Di \sim' b \in Dj$ and $b \in Dj \sim' c \in Dk$, i.e. there exists morphisms $f_1 : i \to w$, $f_2 : j \to w$ in $\mathcal{I}$ for $a \sim' b$, and $g_1 : j \to x$, $g_2 : k \to x$ in $\mathcal{I}$ for $b \sim' c$, such that $Df_1(a) = Df_2(b) \in Dw$ and $Dg_1(b) = Dg_2(c) \in Dx$. We want to show that $a \in Di \sim' c \in Dk$. Since $\mathcal{I}$ is filtered, use (**1.5.14**(ii)) to find morphisms $\alpha_1 : w \to y$ and $\alpha_2 : x \to y$ for an object y in $\mathcal{I}$. Use (**1.5.14**(iii)) to imply that there exists $\beta : y \to z$ in $\mathcal{I}$ such that $\beta \circ \alpha_1 \circ f_2 = \beta \circ \alpha_2 \circ g_1$. We claim that $D(\beta \circ \alpha_1 \circ f_1)(a) = D(\beta \circ \alpha_2 \circ g_2)(c)$:

$$\begin{aligned} D(\beta \circ \alpha_1 \circ f_1)(a) &= D(\beta \circ \alpha_1)(Df_1(a)) = D(\beta \circ \alpha_1)(Df_2(b)), \\ &= D(\beta \circ \alpha_1 \circ f_2)(b) = D(\beta \circ \alpha_2 \circ g_1)(b), \\ &= D(\beta \circ \alpha_2)(Dg_1(b)) = D(\beta \circ \alpha_2)(Dg_2(c)), \\ &= D(\beta \circ \alpha_2 \circ g_2)(c). \end{aligned}$$

For $\sim$ and $\sim'$ being the same: Given $a \in Di \sim b \in Dj$, use the explicit description of $\sim$ found in (**I.5.2.1** [Mor20]), and recursively apply properties (**1.5.14**(ii)) and (**1.5.14**(iii)) of $\mathcal{I}$ to find morphisms $f : i \to k$ and $g : j \to k$ in $\mathcal{I}$, such that $Df(a) = Dg(b) \in Dk$. Thus $a \in Di \sim' b \in Dj$.

Given $a \in Di \sim' b \in Dj$, i.e. there exists $f : i \to k$ and $g : j \to k$ in $\mathcal{I}$ such that $Df(a) = Dg(b) \in Dk$. We have that $a \in Di \sim Dg(b) = Df(a) \in Dk$ and $Dg(b) = Df(a) \in Dk \sim b \in Dj$. Then $a \in Di \sim b \in Dj$ follows from transitivity. □

1.5.17 Lemma (Forgetful Functors for Modules) (I.5.6.3, A.2.4 [Mor20]**):** Let R be a $\mathcal{U}$-ring.

(a) The forgetful functors $\text{For} : {}_R\text{Mod}_\mathcal{U} \to \text{Set}_\mathcal{U}$ and $\text{For} : \text{Mod}_{R\mathcal{U}} \to \text{Set}_\mathcal{U}$ commute with $\mathcal{U}$-small limits. More explicitly, for a $\mathcal{U}$-small category $\mathcal{I}$ and a diagram $D : \mathcal{I} \to {}_R\text{Mod}_\mathcal{U}$ (or $D : \mathcal{I} \to \text{Mod}_{R\mathcal{U}}$), we have that there exists a $\mathcal{U}$-left-R-module structure (or $\mathcal{U}$-right-R-module structure) on $\lim_\mathcal{I}(\text{For} \circ D)$, unique up to isomorphism and given as $(\lim_\mathcal{I}(\text{For} \circ D), 0, +, \cdot)$, such that $(\lim_\mathcal{I}(\text{For} \circ D), 0, +, \cdot) \cong \lim_\mathcal{I} D$ as $\mathcal{U}$-left-R-modules (or $\mathcal{U}$-right-R-modules).

(b) The forgetful functors $\text{For} : {}_R\text{Mod}_\mathcal{U} \to \text{Set}_\mathcal{U}$ and $\text{For} : \text{Mod}_{R\mathcal{U}} \to \text{Set}_\mathcal{U}$ commute with $\mathcal{U}$-small filtered colimits. More explicitly, for a $\mathcal{U}$-small filtered index category $\mathcal{I}$ and a diagram $D : \mathcal{I} \to {}_R\text{Mod}_\mathcal{U}$ (or $D : \mathcal{I} \to \text{Mod}_{R\mathcal{U}}$), we have that there exists an up to isomorphism unique $\mathcal{U}$-left-R-module structure (or $\mathcal{U}$-right-R-module structure) on $\text{colim}_\mathcal{I}(\text{For} \circ D)$, given as $(\text{colim}_\mathcal{I}(\text{For} \circ D), +, \cdot)$, such that $(\text{colim}_\mathcal{I}(\text{For} \circ D), +, \cdot) \cong \text{colim}_\mathcal{I} D$ as $\mathcal{U}$-left-R-modules (or $\mathcal{U}$-right-R-modules).

Proof: For (a): Let $D : \mathcal{I} \to \text{Set}_\mathcal{U}$ be a $\mathcal{U}$-small diagram. The claim follows when comparing the limits found in (**1.5.11**(a)) and (**1.5.12**(a)).

For (b): See references. □

2 Additive and Abelian Categories

We now introduce additive and abelian categories with their related lemmas, along with more universal properties and constructions. Let $\mathcal{U}$ be a Grothendieck universe.

2.1 Additive Categories

2.1.1 Definition (Pre-Additive Categories) (4.1 [Bö20]**):** $\mathcal{A}$ is a *pre-additive $\mathcal{U}$-category* if $\mathcal{A}$ is a $\mathcal{U}$-category and if:

(i) For objects A and B in $\mathcal{A}$, we have an abelian $\mathcal{U}$-group given by $(\text{Hom}_\mathcal{A}(A, B), 0_{A,B}, +_{A,B})$.

(ii) For all objects A, B and C in $\mathcal{A}$, the composition $\circ$ is bilinear:

$$\text{Hom}_\mathcal{A}(A, B) \times \text{Hom}_\mathcal{A}(B, C) \to \text{Hom}_\mathcal{A}(A, C), (f, g) \mapsto (g \circ f).$$

2.1.2 Examples (Pre-Additive Categories) (II.1.1.4 [Mor20]**):**

(a) For a $\mathcal{U}$-ring R and a $\mathcal{U}$-field K, the category of $\mathcal{U}$-vector spaces $\text{VEC}_{K\mathcal{U}}$, as well as ${}_R\text{MOD}_\mathcal{U}$ and $\text{MOD}_{R\mathcal{U}}$, are pre-additive $\mathcal{U}$-categories. This is clear since the space of linear functions from one module to another module forms an abelian $\mathcal{U}$-group. In particular, composition $\circ$ is clearly bilinear.

(b) If $\mathcal{I}$ is a $\mathcal{U}$-small category and $\mathcal{A}$ is a pre-additive $\mathcal{U}$-category, we already know due to (**1.3.4**(b)) that $\text{FUNC}(\mathcal{I}, \mathcal{A})$ is a $\mathcal{U}$-category. It is also easy to check that $\text{FUNC}(\mathcal{I}, \mathcal{A})$ is a pre-additive $\mathcal{U}$-category.

2.1.3 Lemma (Zero Objects) (09SE [JC21]**):** For a pre-additive $\mathcal{U}$-category $\mathcal{A}$, let A be an object of $\mathcal{A}$, then the following are equivalent:

(a) A is an initial object.

(b) A is a final object.

(c) $1_A = 0_{A,A}$ in $\text{Hom}_\mathcal{A}(A, A)$.

Proof: For (a) or (b) implies (c): If A is an initial or terminal object, then $\text{Hom}_\mathcal{A}(A, A)$ can only contain one element and thus $1_A = 0_{A,A}$.

For (c) implies (a): Let $f : A \to W$ be a morphism in $\mathcal{A}$. Then since $f = f \circ 1_A = f \circ 0_{A,A} = 0_{A,A}$ due to the bilinearity of composition, we have that there exists only one element in $\text{Hom}_\mathcal{A}(A, W)$. Thus A is initial.

For (c) implies (b): Let $f : W \to A$ be a morphism in $\mathcal{A}$. Then since $f = 1_A \circ f = 0_{A,A} \circ f = 0_{A,A}$, we have that there exists only one element in $\text{Hom}_\mathcal{A}(A, W)$. Thus A is final. □

2.1.4 Lemma (Biproducts) (II.1.1.6 [Mor20]**):** For a pre-additive $\mathcal{U}$-category $\mathcal{A}$, let $A_1, \ldots, A_n$ be objects in $\mathcal{A}$, then the following are equivalent:

(a) The product $\prod_{i=1}^n A_i$ exists in $\mathcal{A}$.

(b) The coproduct $\coprod_{i=1}^n A_i$ exists in $\mathcal{A}$.

(c) There exists an object W in $\mathcal{A}$ with morphisms $(p_i : A_i \to W)_{i=1,\ldots,n}$ and $(\iota_i : W \to A_i)_{i=1,\ldots,n}$ such that:

(i) $p_i \circ \iota_i = 1_{A_i}$ for all $i = 1, \ldots, n$.

(ii) For $i, j = 1, \ldots, n$, $i \neq j$ we have $p_j \circ \iota_i = 0_{A_i,A_j}$.

(iii) $\iota_1 \circ p_1 + \ldots + \iota_n \circ p_n = 1_W$.

If any of the conditions (a), (b) or (c) are fulfilled, we have that $\prod_{i=1}^n A_i \cong \coprod_{i=1}^n A_i \cong W$ are isomorphic to each other in $\mathcal{A}$. We write $W = \bigoplus_{i=1}^n A_i$ and we have that $(W, (p_i)_{i=1,\ldots n})$ is a product of $A_1, \ldots, A_n$, $(W, (\iota_i)_{i=1,\ldots n})$ is a coproduct of $A_1, \ldots, A_n$. In this case we call $(W, (p_i)_{i=1,\ldots n}, (\iota_i)_{i=1,\ldots n})$ a *biproduct* of $A_1, \ldots, A_n$ in $\mathcal{A}$.

Proof: See reference. □

2.1.5 Definition (Additive Categories) (II.1.2.1 [Mor20]**):** $\mathcal{A}$ is an *additive $\mathcal{U}$-category* if it is a pre-additive $\mathcal{U}$-category that has finite biproducts. Using (**2.1.3**) and (**2.1.4**) we can restate the condition of having finite biproducts as:

(i) For objects $A_1, \ldots A_n$ in $\mathcal{A}$ there exists a biproduct $(\bigoplus_{i=1}^n A_i, (p_i)_{i=1,\ldots n}, (\iota_i)_{i=1,\ldots n})$ in $\mathcal{A}$.

(ii) There exists a zero object 0 in $\mathcal{A}$, since 0 is an finite biproduct indexed over the empty set.

2.1.6 Examples (Additive Categories): For a $\mathcal{U}$-ring R and a $\mathcal{U}$-field K, the categories $\text{VEC}_{K\mathcal{U}}$, ${}_R\text{MOD}_\mathcal{U}$ and $\text{MOD}_{R\mathcal{U}}$ are pre-additive $\mathcal{U}$-categories due to (**2.1.2**(a)). They are furthermore additive $\mathcal{U}$-categories since finite biproducts of modules exist due to (**1.5.9**(b)).

2.1.7 Definition (Additive Functors) (II.1.2.1 [Mor20]**):** A functor between pre-additive $\mathcal{U}$-categories $F : \mathcal{A} \to \mathcal{B}$ is called *additive* if for all $A, B \in \mathrm{Ob}(\mathcal{A})$, the function $\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{B}}(FA, FB), f \mapsto Ff$ is a group homomorphism.

An easy result to check is that if $\mathcal{A}$ and $\mathcal{B}$ have zero objects, then F must send zero objects to zero objects due to (**2.1.3**).

2.1.8 Examples (Additive Functors):

(a) Let $\mathcal{A}$ be an additive $\mathcal{U}$-category, then for an object A in $\mathcal{A}$, the hom-functors $\mathrm{Hom}_{\mathcal{A}}(A, \_)$ and $\mathrm{Hom}_{\mathcal{A}}(\_, A)$ are additive due to the bilinearity of composition.

(b) For a fixed $\mathcal{U}$-small index category $\mathcal{I}$ and a pre-additive $\mathcal{U}$-category $\mathcal{A}$ that contains all limits (respectively colimits) of the diagram $\mathcal{I}$, we have seen due to (**1.5.7**) that $\lim_{\mathcal{I}} : \textsc{Func}(\mathcal{I}, \mathcal{A}) \to \mathcal{A}$ (respectively $\mathrm{colim}_{\mathcal{I}}$) define functors. With $\mathcal{A}$ and $\textsc{Func}(\mathcal{I}, \mathcal{A})$ being pre-additive $\mathcal{U}$-categories due to (**2.1.2**(b)), we have that $\lim_{\mathcal{I}}$ and $\mathrm{colim}_{\mathcal{I}}$ are additive functors.

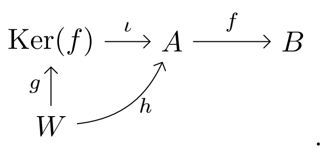
Explanation: Let $u, v : D \to E$ be natural transformations between diagrams $D, E : \mathcal{I} \to \mathcal{A}$ and let $\lim_{\mathcal{I}} u, \lim_{\mathcal{I}} v : (\lim_{\mathcal{I}} D, \alpha) \to (\lim_{\mathcal{I}} E, \beta)$ be the induced morphisms from cones α to β, i.e. the morphisms that factorize to give $u \circ \alpha = \beta \circ \triangle(\lim_{\mathcal{I}} u)$ and $v \circ \alpha = \beta \circ \triangle(\lim_{\mathcal{I}} v)$. $\lim_{\mathcal{I}} u + \lim_{\mathcal{I}} v$ thus helps factorize $(u + v) \circ \alpha = \beta \circ \triangle(\lim_{\mathcal{I}} u + \lim_{\mathcal{I}} v)$ (due to the bilinearity of composition $\circ$) and therefore $\lim_{\mathcal{I}} u + \lim_{\mathcal{I}} v = \lim_{\mathcal{I}}(u + v)$. Thus $\lim_{\mathcal{I}}$ is an additive functor, due to duality $\mathrm{colim}_{\mathcal{I}}$ is also an additive functor.

2.2 Universal Properties

There are many universal properties one may construct, we have seen initial objects, products and their duals in (**1.5.8**). Some more are needed for *Mitchell's embedding theorem.*

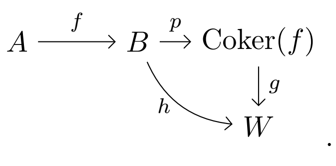
2.2.1 Definitions (Kernels and Cokernels) (0106 [JC21]**), (II.1.3.1** [Mor20]**):** Let $\mathcal{A}$ be a pre-additive $\mathcal{U}$-category. These constructions mimic kernels and cokernels as we know them from modules and vector spaces. Let $f : A \to B$ be a morphism in $\mathcal{A}$.

(a) **Kernels:** The *kernel* of f is $(\mathrm{Ker}(f), \iota)$ where $\mathrm{Ker}(f) \in \mathrm{Ob}(\mathcal{A})$ and $\iota : \mathrm{Ker}(f) \to A$ is a morphism in $\mathcal{A}$ such that $f \circ \iota = 0_{\mathrm{Ker}(f),B}$ and for all $h : W \to A$ whereby $f \circ h = 0_{W,B}$, there exists exactly one morphism $g : W \to \mathrm{Ker}(f)$ in $\mathcal{A}$ such that the diagram commutes:



$(\mathrm{Ker}(f), \iota)$ can be formally defined as the limit of the diagram $A \overset{f}{\underset{0}{\rightrightarrows}} B$.

(b) **Cokernels:** The *cokernel* of f is $(\mathrm{Coker}(f), p)$, where $\mathrm{Coker}(f) \in \mathrm{Ob}(\mathcal{A})$ and $p : B \to \mathrm{Coker}(f)$ is a morphism in $\mathcal{A}$ such that $p \circ f = 0_{A,\mathrm{Coker}(f)}$ and for all $h : B \to W$ whereby $h \circ f = 0_{A,W}$, there exists exactly one morphism $g : \mathrm{Coker}(f) \to W$ in $\mathcal{A}$ such that the diagram commutes:



$(\mathrm{Coker}(f), p)$ can be formally defined as the colimit of a the diagram $A \overset{f}{\underset{0}{\rightrightarrows}} B$.

2.2.2 Examples (Kernels and Cokernels): For categories of abelian $\mathcal{U}$-groups, $\mathcal{U}$-rings, $\mathcal{U}$-modules and $\mathcal{U}$-vector spaces, the definitions of category-theoretic kernels and cokernels coincide with the classical definitions of kernels and cokernels of these algebraic structures.

2.2.3 Definitions (Images and Coimages) (II.1.3.2 [Mor20]**):** For a pre-additive $\mathcal{U}$-category $\mathcal{A}$ and a morphism $f : A \to B$ in $\mathcal{A}$, the *image* $\mathrm{Im}(f)$ is the kernel of the cokernel $p : B \to \mathrm{Coker}(f)$. The *coimage* $\mathrm{Coim}(f)$ is the cokernel of the kernel $\iota : \mathrm{Ker}(f) \to A$.

2.2.4 Lemma (Properties of Kernels and Cokernels) (II.1.3.3 [Mor20]**):** Let $f : A \to B$ be a morphism in an additive $\mathcal{U}$-category $\mathcal{A}$. The following statements are true:

(a) If the kernel of f exists, $\iota : \mathrm{Ker}(f) \to A$ is a monomorphism.

(b) If the cokernel of f exists, $p : \mathrm{Ker}(f) \to A$ is an epimorphism.

(c) If the kernel of f exists, f is a monomorphism if and only if $\mathrm{Ker}(f) = 0$. Furthermore if the coimage of f exists, f is a monomorphism if and only if $\mathrm{Coim}(f) = A$.

(d) If the cokernel of f exists, f is an epimorphism if and only if $\mathrm{Coker}(f) = 0$. Furthermore if the image of f exists, f is an epimorphism if and only if $\mathrm{Im}(f) = B$.

Proof: See reference. □

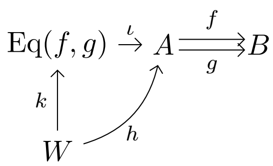
2.2.5 Note (Decompositions of Morphisms) (II.1.3.4 [Mor20]**):** If an additive $\mathcal{U}$-category $\mathcal{A}$ has kernels and cokernels for all its morphisms, then it must also have images and coimages. Applying the universal property of kernels and cokernels also proves that in $\mathcal{A}$, all morphisms $f : A \to B$ have a unique decomposition, depending on the choice of kernel and cokernel, of the form:

$$A \xrightarrow{t_f} \mathrm{Coim}(f) \xrightarrow{u_f} \mathrm{Im}(f) \xrightarrow{v_f} B.$$

See reference for a proof.

2.2.6 Definitions (Equalizers and Coequalizers) (6.2.13, 6.3.9 [Bra16]**):** These may be seen as a generalization of kernels and cokernels. Let $\mathcal{C}$ be a $\mathcal{U}$-category and $f, g : A \to B$ be morphisms in $\mathcal{C}$.

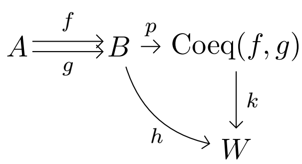
(a) **Equalizers:** The *equalizer* of f and g $(\mathrm{Eq}(f,g), \iota)$ consists of $\mathrm{Eq}(f,g) \in \mathrm{Ob}(\mathcal{C})$ and a morphism $\iota : \mathrm{Eq}(f,g) \to A$, so that for all $h : W \to A$ such that $f \circ h = g \circ h$, there exists exactly one morphism k in $\mathcal{C}$ such that the diagram commutes:



.

The equalizer $(\mathrm{Eq}(f,g), \iota)$ is the limit of the diagram $A \overset{f}{\underset{g}{\rightrightarrows}} B$.

(b) **Coequalizers:** The *coequalizer* of f and g $(\mathrm{Coeq}(f,g), p)$ consists of $\mathrm{Coeq}(f,g) \in \mathrm{Ob}(\mathcal{C})$ and a morphism $p : B \to \mathrm{Coeq}(f,g)$, so that for all $h : B \to W$ such that $h \circ f = h \circ g$, there exists exactly one morphism k in $\mathcal{C}$ such that the diagram commutes:

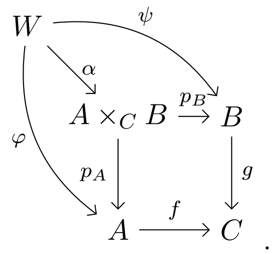


.

The coequalizer $(\mathrm{Coeq}(f,g), \iota)$ is the colimit of the diagram $A \overset{f}{\underset{g}{\rightrightarrows}} B$.

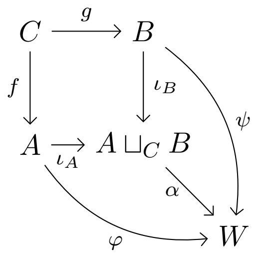
2.2.7 Definitions (Fiber Products and Fiber Coproducts) (6.2.16, 6.3.16 [Bra16]**):** Let $\mathcal{C}$ be a $\mathcal{U}$-category. Fiber products and coproducts may be seen as a generalization of products and coproducts.

(a) **Fiber Products or Pullbacks:** Let $f : A \to C$ and $g : B \to C$ be morphisms in $\mathcal{C}$. The *fiber product* $(A \times_C B, p_A, p_B)$ consists of $A \times_C B \in \mathrm{Ob}(\mathcal{C})$ with projection morphisms $p_A : A \times_C B \to A$ and $p_B : A \times_C B \to A$ such that for all morphisms $\varphi : W \to A$ and $\psi : W \to B$ whereby $f \circ \varphi = g \circ \psi$, there exists exactly a morphism α such that the following diagram commutes:



The fiber product $(A \times_C B, p_A, p_B)$ is the limit of the diagram $A \xrightarrow{f} C \xleftarrow{g} B$.

(b) **Fiber Coproducts or Pushouts:** Let $f : C \to A$ and $g : C \to B$ be morphisms in $\mathcal{C}$. The *fiber coproduct* $(A \sqcup_C B, \iota_A, \iota_B)$ consists of $A \sqcup_C B \in \mathrm{Ob}(\mathcal{C})$ with inclusion morphisms $\iota_A : A \to A \sqcup_C B$ and $\iota_B : B \to A \sqcup_C B$ such that for all morphisms $\varphi : A \to W$ and $\psi : B \to W$ whereby $\varphi \circ f = \psi \circ g$, there exists exactly a morphism α such that the following diagram commutes:



The fiber coproduct $(A \sqcup_C B, \iota_A, \iota_B)$ is the colimit of the diagram $A \xleftarrow{f} C \xrightarrow{g} B$.

2.2.8 Example (Fiber Products): In $\mathrm{Set}_{\mathcal{U}}$, the fiber product can be explicitly constructed. For functions $f : A \to C$ and $g : B \to C$, the set $A \times_C B$ is given explicitly as:

$$A \times_C B = \{(a, b) \in A \times B | f(a) = g(b)\}.$$

2.3 Abelian Categories

Abelian categories generalize desirable properties of categories such as $\mathrm{Ab}_{\mathcal{U}}$, ${}_R\mathrm{Mod}_{\mathcal{U}}$ and $\mathrm{Mod}_{R\mathcal{U}}$.

2.3.1 Definition (Abelian Categories) (II.2.1.1 [Mor20]**):** An additive $\mathcal{U}$-category $\mathcal{A}$ from (**2.1.5**) is an *abelian $\mathcal{U}$-category*, when:

(i) $\mathcal{A}$ contains kernels and cokernels for every morphism $f : A \to B$ in $\mathcal{A}$.

(ii) For the decomposition of morphisms $f : A \to B$ in $\mathcal{A}$ given by (**2.2.5**):

$$A \xrightarrow{t_f} \mathrm{Coim}(f) \xrightarrow{u_f} \mathrm{Im}(f) \xrightarrow{v_f} B,$$

we have that u_f is an isomorphism in $\mathcal{A}$.

2.3.2 Examples (Abelian Categories) (II.2.1.3 [Mor20]**):**

(a) The category of abelian $\mathcal{U}$-groups $\mathrm{Ab}_{\mathcal{U}}$ is an abelian $\mathcal{U}$-category.

(b) For a commutative $\mathcal{U}$-ring R, the categories ${}_R\mathrm{Mor}_{\mathcal{U}}$ and $\mathrm{Mor}_{R\mathcal{U}}$ are abelian $\mathcal{U}$-categories.

(c) If $\mathcal{I}$ is a $\mathcal{U}$-small category and $\mathcal{A}$ is an abelian $\mathcal{U}$-category, it is easy to check that $\textsc{Func}(\mathcal{I}, \mathcal{A})$ is also an additive $\mathcal{U}$-category using (**2.1.2**(b)) and the following: Finite biproducts in $\textsc{Func}(\mathcal{I}, \mathcal{A})$ exist since limits in $\textsc{Func}(\mathcal{I}, \mathcal{A})$ are determined object-wise due to (**I.5.3.1** [Mor20]), and exist since finite biproducts exist in $\mathcal{A}$.

$\textsc{Func}(\mathcal{I}, \mathcal{A})$ is also an abelian $\mathcal{U}$-category, since kernels and cokernels are determined object-wise using (**I.5.3.1** [Mor20]) and the fact that $\mathcal{A}$ is an abelian $\mathcal{U}$-category. Furthermore, the decompositions as seen in (**2.2.5**) still have an isomorphism in the middle, since isomorphisms of natural transformations can be determined object-wise using (**1.3.4**(a)) and the fact that $\mathcal{A}$ is an abelian $\mathcal{U}$-category.

(d) The category of $\mathcal{U}$-rings $\textsc{Rng}_{\mathcal{U}}$ is not an abelian $\mathcal{U}$-category. An initial object of $\textsc{Rng}_{\mathcal{U}}$ is $\mathbb{Z}$ and a final object of $\textsc{Rng}_{\mathcal{U}}$ is the trivial ring 0. If $\textsc{Rng}_{\mathcal{U}}$ were abelian (or even merely preadditive), then $\mathbb{Z}$ and 0 would be zero objects of $\textsc{Rng}_{\mathcal{U}}$ due to (**2.1.3**). Since universal properties are unique up to isomorphism, $\mathbb{Z}$ and 0 would have to be ring-isomorphic to each other, which is not the case and thus leads to a contradiction.

2.3.3 Note (Duals of Abelian Categories): Often very useful in proofs is the fact that for any abelian $\mathcal{U}$-category $\mathcal{A}$, $\mathcal{A}^{\mathrm{op}}$ is also an abelian category.

2.3.4 Lemma (Isomorphisms in Abelian Categories): Let $\mathcal{A}$ be an abelian $\mathcal{U}$-category and $f : A \to B$ be a morphism in $\mathcal{A}$. Then f is an isomorphism if and only if it is a monomorphism and an epimorphism.

Proof: For forward implication: As f is an isomorphism, it must have an inverse morphism g. With the help of g, it is clear that f is left-cancellable and right-cancellable (in terms of composition), which makes f a monomorphism and an epimorphism.

For backward implication: Due to (**2.2.4**(c)), the kernel of f is the zero morphism $\iota : 0 \to A$. It is then easy to see that the coimage of f the identity $1_A : A \to A = \mathrm{Coim}(f)$. Analogously with (**2.2.4**(d)), the cokernel of f is also a zero morphism which makes $1_B : B = \mathrm{Im}(f) \to B$ the image of f. Then using the decomposition from (**2.2.5**), it is clear that $f = u_f$ is an isomorphism. ☐

2.3.5 Note (Abelian Categories have Finite Limits and Colimits) (010D [JC21]): In the reference, it is proven that since $\mathcal{A}$ contains all finite products and coproducts, equalizers and coequalizers, $\mathcal{A}$ also must have all finite limits and colimits.

2.4 Exact Functors and Short Exact Sequences

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian $\mathcal{U}$-categories.

2.4.1 Note (Image-Kernel Morphisms) (II.2.1.7 [Mor20]): Let $f : A \to B$ and $g : B \to C$ be two morphisms in $\mathcal{A}$ such that $g \circ f = 0_{A,C}$, then the universal properties of kernels and cokernels can be used to induce a canonical morphism $\theta : \mathrm{Im}(f) \to \mathrm{Ker}(g)$ such that for the kernel morphism $\iota : \mathrm{Ker}(g) \to B$ and the decomposition morphisms u_f and t_f of f seen in (**2.2.5**), we have $f = v_f \circ u_f \circ t_f = \iota \circ \theta \circ u_f \circ t_f$. Due to $u_f \circ t_f$ being an epimorphism, we have that $v_f = \iota \circ \theta$ is a monomorphism, implying that θ is a monomorphism.

2.4.2 Definition (Exact Sequences) (II.2.1.8 [Mor20]):

(a) A sequence $\ldots \to A \to B \to \ldots$ of any length is an *exact sequence* if the image of the preceding morphism is isomorphic to the kernel of the following morphism via the induced morphism θ from (**2.4.1**).

(b) A diagram of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ is called a *short exact sequence* if the image of the preceding morphism is isomorphic to the kernel of the following morphism via the induced morphism θ from (**2.4.1**). Due to (**2.2.4**), this is equivalent to f being a monomorphism, g being an epimorphism and $\mathrm{Im}(f) \cong \mathrm{Ker}(g)$ via θ from (**2.4.1**).

2.4.3 Note (Short Exact Sequences): It is clear that $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if only if its dual is exact in $\mathcal{A}^{\text{op}}$.

2.4.4 Example (Short Exact Sequences): For a monomorphism $f : A \to B$ in $\mathcal{A}$, we have that:

$$0 = \text{Ker}(f) \to A \xrightarrow{f} B \to \text{Coker}(f) \to 0,$$

is a short exact sequence, this follows directly from (**2.2.4**(c)).

2.4.5 Lemma (Split Short Exact Sequences) (II.2.1.11 [Mor20]**), (4.1.8** [Sch21]**):** Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in $\mathcal{A}$, then the following conditions are equivalent:

(i) There exists a morphism $i : B \to A$ such that $i \circ f = 1_A$.

(ii) There exists a morphism $h : C \to B$ such that $g \circ h = 1_C$.

(iii) There exists morphisms $i : B \to A$ and $h : C \to B$ such that: $\varphi : B \to A \oplus C$ induced by i and g, $\psi : A \oplus C \to B$ induced by f and h, are inverse to each other, and thus $B \cong A \oplus C$.

If one of the conditions are fulfilled, then $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is said to be *split*. Due to (i) and (ii), the existence of h is guaranteed given i and vice versa.

Proof: See references. □

2.4.6 Definitions (Exact Functors) (II.2.3.1 [Mor20]**), (4.2.1** [Sch21]**):** Due to (**2.3.5**), $\mathcal{A}$ and $\mathcal{B}$ contain all finite limits and colimits. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor, then:

(a) F is *left exact* if F commutes with finite limits i.e. for any diagram $D : \mathcal{I} \to \mathcal{A}$ such that $\mathcal{I}$ is finite, we have $F(\lim_{\mathcal{I}} D) \cong \lim_{\mathcal{I}} FD$.

(b) F is *right exact* if F commutes with finite colimits i.e. for any diagram $D : \mathcal{I} \to \mathcal{A}$ such that $\mathcal{I}$ is finite, we have $F(\text{colim}_{\mathcal{I}} D) \cong \text{colim}_{\mathcal{I}} FD$.

(c) F is *exact* if it is left exact and right exact.

2.4.7 Notes (Exact Functors): Due to duality, the following equivalences are clear:

(a) $F : \mathcal{A} \to \mathcal{B}$ is left exact if and only if its opposite functor $F^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathcal{B}^{\text{op}}$ is right exact.

(b) $F : \mathcal{A} \to \mathcal{B}$ is exact if and only if its opposite functor $F^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathcal{B}^{\text{op}}$ is exact.

The following lemma is very useful for characterizing exact functors.

2.4.8 Lemma (Equivalent Definitions of Exact Functors) (II.2.3.2 [Mor20]**), (4.2.2** [Sch21]**):** For an additive functor $F : \mathcal{A} \to \mathcal{B}$ the following statements are equivalent:

(i) F is left exact.

(ii) F commutes with kernels, i.e. $F(\text{Ker}(f)) \cong \text{Ker}(Ff)$.

(iii) For any exact sequence $0 \to A \to B \to C$, the sequence $0 \to FA \to FB \to FC$ is also exact.

The duals of these statements also apply for F being right exact, i.e. the following statements are equivalent:

(iv) F is right exact.

(v) F commutes with cokernels, i.e. $F(\text{Coker}(f)) \cong \text{Coker}(Ff)$.

(vi) For any exact sequence $A \to B \to C \to 0$, the sequence $FA \to FB \to FC \to 0$ is also exact.

Combining the previous two sets of statements, the following statements are equivalent:

(vii) F is exact.

(viii) F commutes with kernels and cokernels, i.e. $F(\mathrm{Ker}(f)) \cong \mathrm{Ker}(Ff)$, $F(\mathrm{Coker}(f)) \cong \mathrm{Coker}(Ff)$.

(ix) For any exact sequence $0 \to A \to B \to C \to 0$, the sequence $0 \to FA \to FB \to FC \to 0$ is also exact.

Proof: See references. □

2.4.9 Note (Hom-Functors are Left Exact) (II.2.3.4 [Mor20]**):** Let A be an object in $\mathcal{A}$, it is clear that $\mathrm{Hom}_{\mathcal{A}}(A, \_) : \mathcal{A} \to \mathrm{AB}_{\mathcal{U}}$ and $\mathrm{Hom}_{\mathcal{A}}(\_, A) : \mathcal{A}^{\mathrm{op}} \to \mathrm{AB}_{\mathcal{U}}$ are additive functors. Due to $\mathcal{A}$ containing finite limits (**2.3.5**), due to the properties of hom-functors from (**1.5.10**) and due to the forgetful functor $\mathrm{For} : \mathrm{AB}_{\mathcal{U}} \to \mathrm{SET}_{\mathcal{U}}$ commuting with $\mathcal{U}$-small limits due to (**1.5.17**(a)), we have that $\mathrm{Hom}_{\mathcal{A}}(A, \_) : \mathcal{A} \to \mathrm{AB}_{\mathcal{U}}$ and $\mathrm{Hom}_{\mathcal{A}}(\_, A) : \mathcal{A}^{\mathrm{op}} \to \mathrm{AB}_{\mathcal{U}}$ both commute with finite limits and thus they are left exact.

2.4.10 Lemma (Adjoint Functors) (II.2.3.3 [Mor20]**):** For additive functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{A} \to \mathcal{B}$ such that (F, G) is an adjoint pair of functors, F is right exact and G is left exact.

Proof: $\mathcal{A}$ and $\mathcal{B}$ have all finite limits and colimits due to (**2.3.5**). For a finite index category $\mathcal{I}$ and a diagram $D : \mathcal{I} \to \mathcal{A}$, we want to check that $(G(\lim_{\mathcal{I}} D), G \circ \alpha) \cong (\lim_{\mathcal{I}}(G \circ D), \beta)$, whereby $(\lim_{\mathcal{I}} D, \alpha)$ and $(\lim_{\mathcal{I}}(G \circ D), \beta)$ are limit cones. As $(\lim_{\mathcal{I}}(G \circ D), G \circ \alpha)$ is a $G \circ D$-cone, we have a unique morphism $\varphi : G(\lim_{\mathcal{I}} D) \to \lim_{\mathcal{I}}(G \circ D)$ which we claim is an isomorphism. Due to the *Yoneda lemma* (**1.4.7**) it is enough to show that the morphism between presheaves $h_\varphi : \mathrm{Hom}_{\mathcal{B}}(\_, G(\lim_{\mathcal{I}} D)) \to \mathrm{Hom}_{\mathcal{B}}(\_, \lim_{\mathcal{I}}(G \circ D))$ is an isomorphism, we have:

$$\mathrm{Hom}_{\mathcal{B}}(\_, G(\lim_{\mathcal{I}} D)) \cong \mathrm{Hom}_{\mathcal{B}}(F(\_), \lim_{\mathcal{I}} D)) \cong \lim_{i \in \mathrm{Ob}(\mathcal{I})}(\mathrm{Hom}_{\mathcal{B}}(F(\_), D(i))),$$

with the final isomorphism due to (**2.4.9**). Then with the (F, G) adjunction we have:

$$\mathrm{Hom}_{\mathcal{B}}(\_, G(\lim_{\mathcal{I}} D)) \cong \lim_{i \in \mathrm{Ob}(\mathcal{I})}(\mathrm{Hom}_{\mathcal{B}}(\_, (G \circ D)(i))) \cong \mathrm{Hom}_{\mathcal{B}}(\_, \lim_{\mathcal{I}}(G \circ D)),$$

with the final isomorphism due to (**2.4.9**). Therefore G is left exact as it commutes with finite limits. The case for F being right exact follows analogously due to duality. □

2.4.11 Corollary (Limits and Colimits are Adjoint) (II.2.3.4 [Mor20]**):** For an abelian $\mathcal{U}$-category $\mathcal{A}$. $\lim_{\mathcal{I}} : \mathrm{FUNC}(\mathcal{I}, \mathcal{A}) \to \mathcal{A}$ is left exact and $\mathrm{colim}_{\mathcal{I}} : \mathrm{FUNC}(\mathcal{I}, \mathcal{A}) \to \mathcal{A}$ is right exact.

Proof: $\lim_{\mathcal{I}}$ and $\mathrm{colim}_{\mathcal{I}}$ are additive functors due to (**2.1.8**(b)). Furthermore in (**1.5.5**(a)), we saw that for a $\mathcal{U}$-small diagram $D : \mathcal{I} \to \mathcal{A}$ we have $\mathrm{Hom}_{\mathrm{FUNC}(\mathcal{I},\mathcal{A})}(\triangle(\_), D) \cong \mathrm{Hom}_{\mathcal{C}}(\_, \lim_{\mathcal{I}} D)$. Due to the construction of $\lim_{\mathcal{I}}$ as a functor in (**1.5.7**), we see that $(\triangle, \lim_{\mathcal{I}})$ are an adjoint pair. Then due to (**2.4.10**) we have that $\lim_{\mathcal{I}}$ is left exact. For $\mathrm{colim}_{\mathcal{I}}$ we have analogously that $\mathrm{Hom}_{\mathrm{FUNC}(\mathcal{I},\mathcal{A})}(D, \triangle(\_)) \cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{I}} D, \_)$, which makes $(\mathrm{colim}_{\mathcal{I}}, \triangle)$ an adjoint pair and therefore with (**2.4.10**), $\mathrm{colim}_{\mathcal{I}}$ is right exact. □

2.4.12 Lemma (Filtered Colimits are Exact) (II.2.3.4 [Mor20]**):** Let R be a $\mathcal{U}$-ring and $\mathcal{I}$ be a $\mathcal{U}$-small filtered category. Then the functor $\mathrm{colim}_{\mathcal{I}} : \mathrm{FUNC}(\mathcal{I}, {}_R\mathrm{MOD}_{\mathcal{U}}) \to {}_R\mathrm{MOD}_{\mathcal{U}}$ is exact, analogously $\mathrm{colim}_{\mathcal{I}} : \mathrm{FUNC}(\mathcal{I}, \mathrm{MOD}_{R\mathcal{U}}) \to \mathrm{MOD}_{R\mathcal{U}}$ is also exact.

Proof: Due to (**1.3.9**), we must only prove the case for ${}_R\mathrm{MOD}_{\mathcal{U}}$. ${}_R\mathrm{MOD}_{\mathcal{U}}$ and $\mathrm{SET}_{\mathcal{U}}$ contain all $\mathcal{U}$-small indexed limits and colimits due to (**1.5.11**) and (**1.5.12**), thus (**I.5.3.1** [Mor20]) implies that $\mathrm{FUNC}(\mathcal{I}, {}_R\mathrm{MOD}_{\mathcal{U}})$ and $\mathrm{FUNC}(\mathcal{I}, \mathrm{SET}_{\mathcal{U}})$ contain all $\mathcal{U}$-small limits and colimits. From (**2.4.11**) we already have that $\mathrm{colim}_{\mathcal{I}} : \mathrm{FUNC}(\mathcal{I}, {}_R\mathrm{MOD}_{\mathcal{U}}) \to {}_R\mathrm{MOD}_{\mathcal{U}}$ is right exact so it suffices to show that $\mathrm{colim}_{\mathcal{I}} : \mathrm{FUNC}(\mathcal{I}, {}_R\mathrm{MOD}_{\mathcal{U}}) \to {}_R\mathrm{MOD}_{\mathcal{U}}$ is left exact, i.e. $\mathrm{colim}_{\mathcal{I}}$ commutes with finite limits.

Let $\mathcal{J}$ be a finite index category. We will use the following limit and colimit functors $\lim_{\mathcal{J}} : \mathrm{FUNC}(\mathcal{J}, {}_R\mathrm{MOD}_{\mathcal{U}}) \to {}_R\mathrm{MOD}_{\mathcal{U}}$, $\lim_{\mathcal{J}}^{\mathrm{SET}} : \mathrm{FUNC}(\mathcal{J}, \mathrm{SET}_{\mathcal{U}}) \to \mathrm{SET}_{\mathcal{U}}$ and $\mathrm{colim}_{\mathcal{I}}^{\mathrm{SET}} : \mathrm{FUNC}(\mathcal{I}, \mathrm{SET}_{\mathcal{U}}) \to \mathrm{SET}_{\mathcal{U}}$ along with $\mathrm{colim}_{\mathcal{I}}$. Using the universal properties of limits and colimits, there exists a unique module homomorphism $\varphi : \mathrm{colim}_{\mathcal{I}} \lim_{\mathcal{J}} D(i,j) \to \lim_{\mathcal{J}} \mathrm{colim}_{\mathcal{I}} D(i,j)$.

In (**I.5.6.4** [Mor20]) it is proven that $\operatorname{colim}_{\mathcal{I}}^{\text{SET}} : \text{FUNC}(\mathcal{I}, \text{SET}_{\mathcal{U}}) \to \text{SET}_{\mathcal{U}}$ commutes with finite limits as $\mathcal{I}$ is a $\mathcal{U}$-small filtered category. Then using (**1.5.17**) with the forgetful functor $\text{For} : {}_R\text{MOD}_{\mathcal{U}} \to \text{SET}_{\mathcal{U}}$, we have the following bijections for a diagram $D : \mathcal{I} \times \mathcal{J} \to {}_R\text{MOD}_{\mathcal{U}}$:

$$\begin{aligned} \text{For}(\operatorname{colim}_{\mathcal{I}}\lim_{\mathcal{J}} D(i,j)) &\cong \operatorname{colim}_{\mathcal{I}}^{\text{SET}}\lim_{\mathcal{J}}^{\text{SET}}\text{For}(D(i,j)), \\ \cong \lim_{\mathcal{J}}^{\text{SET}}\operatorname{colim}_{\mathcal{I}}^{\text{SET}}\text{For}(D(i,j)) &\cong \text{For}(\lim_{\mathcal{J}}\operatorname{colim}_{\mathcal{I}} D(i,j)). \end{aligned}$$

As For commutes with $\operatorname{colim}_{\mathcal{I}}$ and $\lim_{\mathcal{J}}$, the isomorphism between $\text{For}(\operatorname{colim}_{\mathcal{I}}\lim_{\mathcal{J}} D(i,j))$ and $\text{For}(\lim_{\mathcal{J}}\operatorname{colim}_{\mathcal{I}} D(i,j))$ is the same as $\text{For}(\varphi)$. Thus φ is a bijective module homomorphism, i.e. an isomorphism, and therefore $\operatorname{colim}_{\mathcal{I}}\lim_{\mathcal{J}} D(i,j) \cong \lim_{\mathcal{J}}\operatorname{colim}_{\mathcal{I}} D(i,j)$. □

2.5 Subobjects and Quotients

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian $\mathcal{U}$-categories. Subobjects and quotients are useful generalizations of subsets and quotients that we know from dealing with sets and groups.

2.5.1 Definitions (Subobjects and Quotients) (II.2.2.1 [Mor20]**):** Let A be an object in $\mathcal{A}$, then:

(a) A *subobject* of A is an element of $\{(B, \iota) | \iota : B \to A \text{ monomorphism in } \mathcal{A}\}/\sim$, where $\sim$ is the equivalence relation: $(B_1, \iota_1) \sim (B_2, \iota_2)$ if and only if there exists an isomorphism $\varphi : B_1 \to B_2$ such that $\iota_1 = \iota_2 \circ \varphi$. We write $B \subset A$ to refer to a certain subobject of A. Let $\text{Sub}(A)$ be the set of subobjects of A.

(b) A *quotient* of A is an element of $\{(B, p) | p : A \to B \text{ epimorphism in } \mathcal{A}\}/\sim$, where $\sim$ is the equivalence relation: $(B_1, p_1) \sim (B_2, p_2)$ if and only if there exists an isomorphism $\varphi : B_1 \to B_2$ such that $p_2 = \varphi \circ p_1$. Let $\text{Quot}(A)$ be the set of quotients of A.

2.5.2 Note (Subobject-Quotient Bijections) (II.2.2.2 [Mor20]**):** Let A be an object of $\mathcal{A}$. For every quotient of A given via the epimorphism $p : A \to B$, we have the kernel $\iota : \text{Ker}(p) \to A$ which is a monomorphism due to (**2.2.4**(c)). Therefore, $(\text{Ker}(p), \iota)$ is a subobject of A which is uniquely determined by (B, p), as the kernel $\iota : \text{Ker}(p) \to A$ is uniquely determined by p up to isomorphism. Furthermore, the cokernel epimorphism $p' : A \to \text{Coker}(\iota) = \text{Coim}(p) \cong \text{Im}(p) \cong B$ defines the same quotient B as $p : A \to B$ did.

Dually, for every subobject of A given as the monomorphism $\iota : C \to A$, the cokernel epimorphism $p : A \to \text{Coker}(\iota)$ uniquely defines a quotient $(\text{Coker}(\iota), p)$ of A and $\iota' : \text{Ker}(p) \to A$ defines the same subobject of A as $\iota : C \to A$.

We have defined a bijection between $\text{Sub}(A)$ and $\text{Quot}(A)$, and thus we can represent a quotient of A, given via the epimorphism $p : A \to B$, uniquely as $A/\text{Ker}(p)$.

2.5.3 Note (Subobject Order Relations) (II.2.2.5 [Mor20]**):** It can be shown that the subobject relation $\subset$ from (**2.5.1**(a)) defines a partial order relation on $\text{Sub}(A)$. Furthermore, it can be shown that this relation forms a *lattice*, i.e. for subobjects B and C of A we have a well-defined maximum $\max(B, C) = B \cup C \in \text{Sub}(A)$ and minimum $\min(B, C) = B \cap C \in \text{Sub}(A)$. This is proven in (**II.2.2.5** [Mor20]).

2.6 Cartesian and Cocartesian Diagrams

Let $\mathcal{A}$ be an abelian $\mathcal{U}$-category and $\mathcal{C}$ be a $\mathcal{U}$-category. Cartesian and cocartesian diagrams are an important construction as they allow us to quickly determine properties of the morphisms in such diagrams, using some clever tricks with universal properties.

2.6.1 Definitions (Cartesian and Cocartesian Diagrams): Observe the diagram of morphisms in $\mathcal{C}$:

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array}.$$

The diagram is called a *cartesian square* if $(B\times_D C, b, a)$ exists as a pullback in $\mathcal{C}$ and $A\cong B\times_D C$, and the diagram is a *cocartesian square* if $(B\sqcup_A C, g, f)$ exists as a pushout in $\mathcal{C}$ and $D\cong B\sqcup_A C$.

2.6.2 Lemma (Cartesian and Cocartesian Diagrams) (II.2.1.15 [Mor20]**):** For a commutative square in $\mathcal{A}$:

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array},$$

we have the following:

(a) The following statements are equivalent:

(i) The induced morphism $\varphi: A\to B\times_D C$ is an epimorphism.

(ii) The induced morphism $\psi: B\sqcup_A C\to D$ is a monomorphism.

(iii) The sequence $A\overset{u}{\to} B\oplus C\overset{v}{\to} D$ is exact, whereby for the inclusions ι_B and ι_C and projections p_B and p_C of $B\oplus C$ we have $u=\iota_B\circ a+\iota_C\circ b$ as the morphism induced by a and b and similarly $v=f\circ p_B-g\circ p_C$.

(b) $0\to A\overset{u}{\to} B\oplus C\overset{v}{\to} D$ is exact if and only if the square is cartesian. $A\overset{u}{\to} B\oplus C\overset{v}{\to} D\to 0$ is exact if and only if the square is cocartesian.

Proof: For (a), (i) implies (iii): We have $v\circ u=(f\circ p_B-g\circ p_C)\circ(\iota_B\circ a+\iota_B\circ c)=f\circ a-g\circ c=0_{A,D}$ due to the commutativity of the square. Due to (**2.4.1**) there exists a canonical monomorphism $\theta:\mathrm{Im}(u)\to\mathrm{Ker}(v)$ which we claim is an isomorphism.

Let $\iota: B\times_D C\to B\oplus C$ be the morphism in $\mathcal{A}$ induced by the projections $p'_B: B\times_D C\to B$ and $p'_C: B\times_D C\to C$. Then for all objects W in $\mathcal{A}$, we have the isomorphisms induced by ι:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(W,B\times_D C)&\cong\{(\alpha:W\to B,\beta:A\to C)|f\circ\alpha=g\circ\beta\},\\&\cong\{\gamma\in\mathrm{Hom}_{\mathcal{A}}(W,B\oplus C)|v\circ\gamma=0\}\cong\mathrm{Hom}_{\mathcal{A}}(W,\mathrm{Ker}(v)).\end{aligned}$$

Thus, it is clear that ι fulfills the universal property of the kernel of v, we have then $\mathrm{Ker}(v)=B\times_D C$. Furthermore, as φ and u are induced from the same morphisms a and b, we have $u=\iota\circ\varphi$ where ι is a monomorphism and φ and epimorphism. It follows that $\mathrm{Im}(u)=\mathrm{Im}(\varphi)=B\times_D C$ due to (**2.2.4**) and thus we have $\theta: B\times_D C\to B\times_D C$.

We have that $\theta=1_{B\times_D C}$ since for the decomposition of $u=v_u\circ u_u\circ t_u$ given by (**2.3.1**), we see that $\iota\circ 1_{B\times_D C}\circ u_u\circ t_u=u$, i.e. $1_{B\times_D C}$ fulfills the definition of θ as seen in (**II.2.1.7** [Mor20]). Thus θ is an isomorphism.

For (a), (iii) implies (ii): Let $\alpha: B\oplus C\to B\oplus C$ be the morphism induced from the morphisms $1_B: B\to B$ and $-1_C: C\to C$, i.e. $\alpha=\iota_B\circ p_B-\iota_C\circ p_C$. It is thus clear that $\alpha\circ\alpha=1_{B\oplus C}$ which makes α an isomorphism. If we let $u'=\alpha\circ u$ and $v'=v\circ\alpha$, then we have $v'\circ u'=v\circ\alpha\circ\alpha\circ u=v\circ u=0$. Due to α being an isomorphism, the morphism $\theta':\mathrm{Im}(u')\to\mathrm{Ker}(v')$ given by (**2.4.1**) is an isomorphism if and only if θ is an isomorphism. Thus, (iii) is equivalent to $A\overset{u'}{\to} B\oplus C\overset{v'}{\to} D$ being exact.

Let $p: B\sqcup_D C\to B\oplus C$ be the epimorphism (as the dual of ι in $\mathcal{A}^{\mathrm{op}}$) in $\mathcal{A}$ induced by the inclusions $\iota'_B: B\to B\sqcup_D C$ and $\iota'_C: C\to B\sqcup_D C$. Then for all objects W in $\mathcal{A}$, we have the isomorphisms induced by p:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(B\sqcup_D C,W)&\cong\{(\alpha:B\to W,\beta:C\to W)|\alpha\circ a=\beta\circ b\},\\&=\{\gamma\in\mathrm{Hom}_{\mathcal{A}}(B\oplus C,W)|\gamma\circ u'=0\}\cong\mathrm{Hom}_{\mathcal{A}}(\mathrm{Coker}(u'),W).\end{aligned}$$

Thus, we have that $B\sqcup_A C=\mathrm{Coker}(u')$. Since $\psi=v'\circ p$ and p is an epimorphism, we have $\mathrm{Coim}(\psi)=\mathrm{Coim}(v'\circ p)=\mathrm{Coim}(v')$. Since $\theta':\mathrm{Im}(u')\to\mathrm{Ker}(v')$ is an isomorphism, we have the following isomorphisms given via (**2.5.2**):

$$B \sqcup_A C = \operatorname{Coker}(u') \cong (B \oplus C)/\operatorname{Im}(u') \cong (B \oplus C)/\operatorname{Ker}(v') \cong \operatorname{Coim}(v').$$

We then have $\operatorname{Coim}(\psi) \cong B \sqcup_A C$ which means that ψ is a monomorphism due to (**2.2.4**(c)).

For (a), (ii) implies (iii): (ii) being true in $\mathcal{A}$ is clearly equivalent to (i) being true in $\mathcal{A}^{\text{op}}$. When applying (i) implies (iii) on $\mathcal{A}^{\text{op}}$, we receive the exactness of the sequence $A \overset{u'^{\text{op}}}{\leftarrow} B \oplus C \overset{v'^{\text{op}}}{\leftarrow} D$ in $\mathcal{A}^{\text{op}}$. This implies the exactness of the sequence $A \overset{v'}{\to} B \oplus C \overset{v'}{\to} D$ in $\mathcal{A}$, which we saw is equivalent to the exactness of $A \overset{u}{\to} B \oplus C \overset{v}{\to} D$.

For (a), (iii) implies (i): As we know that $A \overset{v'}{\to} B \oplus C \overset{v'}{\to} D$ is exact in $\mathcal{A}$, we know that $A \overset{u'^{\text{op}}}{\leftarrow} B \oplus C \overset{v'^{\text{op}}}{\leftarrow} D$ is exact in $\mathcal{A}^{\text{op}}$, where we apply (iii) implies (ii) on $\mathcal{A}^{\text{op}}$. This gives us directly that φ^{op} is a monomorphism in $\mathcal{A}^{\text{op}}$, which implies that φ is an epimorphism in $\mathcal{A}$.

For (b): Due to duality, it is enough to show that $0 \to A \overset{u}{\to} B \oplus C \overset{v}{\to} D$ is exact if and only if the square is cartesian. Due to (a) it is also enough to show that u is a monomorphism if and only if $\varphi : A \to B \times_D C$ is a monomorphism, this is clear since $u = \iota \circ \varphi$ whereby ι is a monomorphism. □

2.6.3 Lemma (Cartesian and Cocartesian Diagrams) (08N4 [JC21]**), (I.7.1** [Mit65]**), (II.2.1.16** [Mor20]**):** For a commutative diagram in $\mathcal{C}$ given by:

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & D \end{array},$$

the following applies if the square is cartesian, i.e. the pullback $(B \times_D C, b, a)$ exists in $\mathcal{C}$ with an isomorphism $\varphi : A \to B \times_D C$:

(a) If f is a monomorphism, so is b a monomorphism.

(b) If $\mathcal{C}$ is an abelian $\mathcal{U}$-category and f is an epimorphism, so is b an epimorphism.

If the square is cocartesian i.e. the pushout $(B \sqcup_A C, g, f)$ exists in $\mathcal{C}$ with an isomorphism $\psi : B \sqcup_A C \to D$, dual statements naturally apply:

(c) If b is an epimorphism, so is f an epimorphism.

(d) If $\mathcal{C}$ is an abelian $\mathcal{U}$-category and b is a monomorphism, so is f a monomorphism.

Proof: It is enough to prove (a) and (d), as (b) and (c) follow from the duality of cartesian and cocartesian squares and (**1.2.11**).

For (a): In order to prove that b is a monomorphism, let $\alpha, \beta : W \to A$ be morphisms in $\mathcal{C}$ such that $b \circ \alpha = b \circ \beta$. Therefore we have:

$$f \circ a \circ \alpha = g \circ b \circ \alpha = g \circ b \circ \beta = f \circ a \circ \beta.$$

As f is a monomorphism, we then have $a \circ \alpha = a \circ \beta$ and $b \circ \alpha = b \circ \beta$. As we have a cartesian square, there exists a unique morphism $\varphi : W \to A$ so that $b \circ \varphi = b \circ \alpha = b \circ \beta$ and $a \circ \varphi = a \circ \alpha = a \circ \beta$ and $g \circ b \circ \varphi = f \circ a \circ \varphi$. Therefore $\alpha = \varphi = \beta$ and b is a monomorphism.

For (d): Observe $B \oplus C$ with the inclusions ι_B and ι_C and projections p_B and p_C. Since b is a monomorphism, we have that $u = \iota_B \circ a + \iota_C \circ b$ is a monomorphism since for all morphisms α and β such that $u \circ \alpha = u \circ \beta$ we have $p_C \circ u \circ \alpha = p_C \circ u \circ \beta$ and thus $b \circ \alpha = b \circ \beta$ and $\alpha = \beta$. Due to (**2.6.2**(b)), the square is also cartesian.

To show that f is a monomorphism, we can show that for a morphism $h : W \to B$ such that $f \circ h = 0_{W,D}$ we have $h = 0_{W,B}$ (due to the bilinearity of composition). We have $f \circ h = g \circ 0_{W,C}$, then the morphisms h and $0_{W,C}$ induce a canonical morphism $l : W \to A$ since the square is cartesian with $A \cong B \times_D C$. Since $b \circ l = 0_{W,C}$ and since b is a monomorphism, we have $l = 0_{W,A}$. Therefore since $0_{W,A}$ is also the canonical morphism induced from zero morphisms $0_{W,B}$ and $0_{W,C}$, we have that $h = 0_{W,B}$ and thus f is a monomorphism. □

2.6.4 Lemma (Cartesian and Cocartesian Diagrams) (II.2.2.6 [Mor20]**):** Let A, B and C be objects of $\mathcal{A}$ such that B and C are subobjects of A, then the following diagram:

$$\begin{array}{ccc} B \cap C & \xrightarrow{a} & B \\ {\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & B \cup C \end{array},$$

with the corresponding monomorphisms, is both cartesian and cocartesian.

As a consequence, for any two morphisms $\alpha : B \to W$ $\beta : C \to W$ in $\mathcal{A}$ such that their *restrictions* on $B \cap C$ are the same, i.e. $\alpha \circ a = \beta \circ b$, there must exist a $\gamma : B \cup C \to W$ that *extends* α and β onto $B \cup C$, i.e. $\alpha = \gamma \circ f$ and $\beta = \gamma \circ g$.

Proof: The following is true for $u : B \cap C \to B \oplus C$ and $v : B \oplus C \to B \cup C$ as seen in (**2.6.2**): u is clearly a monomorphism as a and b are monomorphisms and v is an epimorphism since $\operatorname{Im}(v)$ is a subobject $B \cup C$ for which B and C are subobjects of $\operatorname{Im}(v)$ (which implies $\operatorname{Im}(v) \cong B \cup C$ due to maximality). $v \circ u$ is the zero morphism due to the commutativity of the diagram, which implies $\operatorname{Im}(u) \subset \operatorname{Ker}(v)$ due to (**2.4.1**). If $\operatorname{Ker}(v) \subset \operatorname{Im}(u)$ were not true, i.e. $B \cap C \cong \operatorname{Im}(u) \subsetneq \operatorname{Ker}(v)$, then it would be clear that the kernel morphism of v, $\iota : \operatorname{Ker}(v) \to B \oplus C$, induces monomorphisms $\iota_B : \operatorname{Ker}(v) \to B$ and $\iota_C : \operatorname{Ker}(v) \to C$ such that $B \cap C$ is a subobject of $\operatorname{Ker}(v)$ and $\operatorname{Ker}(v)$ is a subobject of B and C. This contradicts the minimality of $B \cap C$ and thus we have $\operatorname{Ker}(v) \subset \operatorname{Im}(u)$ and thus $\operatorname{Im}(u) \cong \operatorname{Ker}(v)$ via the morphism $\theta : \operatorname{Im}(u) \to \operatorname{Ker}(v)$.

We have shown the exactness of the sequences seen in (**2.6.2**(b)) and thus the diagram is cartesian and cocartesian. □

3 Injectives, Projectives, Generators and Cogenerators

We have established many useful results from category theory and abelian categories. Now we will begin defining structures that will eventually result in proving that the category of modules have enough injectives, as seen in (**3.4.8**), an important result for *Mitchell's embedding theorem.* Let $\mathcal{U}$ be a Grothendieck universe.

3.1 Injective and Projective Objects

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian $\mathcal{U}$-categories.

3.1.1 Definitions (Injective and Projective Objects) (II.2.4.1 [Mor20]**):**

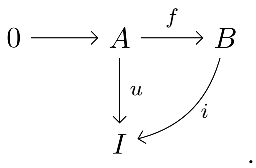
(a) An object I in $\mathcal{A}$ is *injective* when the functor $\operatorname{Hom}_{\mathcal{A}}(\_, I) : \mathcal{A}^{\mathrm{op}} \to \mathrm{AB}_{\mathcal{U}}$ is exact.

(b) An object P in $\mathcal{A}$ is *projective* when the functor $\operatorname{Hom}_{\mathcal{A}}(P, \_) : \mathcal{A} \to \mathrm{AB}_{\mathcal{U}}$ is exact.

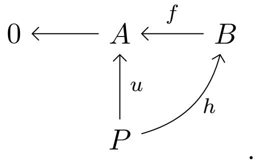
It is clear that these definitions are dual to each other, i.e. I is injective in $\mathcal{A}$ if and only if it is projective in $\mathcal{A}^{\mathrm{op}}$.

3.1.2 Lemma (Equivalent Definitions of Injective and Projective Objects) (5.19, 5.20 [Bö20]**), (II.2.4.2** [Mor20]**):**

(a) An object I in $\mathcal{A}$ is injective if and only if for all monomorphisms $f : A \to B$ and morphisms $u : A \to I$ in $\mathcal{A}$, there exists an $i : B \to I$ (not necessarily unique) so that $i \circ f = u$, i.e. the following diagram commutes:



(b) An object P in $\mathcal{A}$ is projective if and only if for all epimorphisms $f : B \to A$ and morphisms $u : P \to A$ in $\mathcal{A}$, there exists a $h : P \to B$ (not necessarily unique) so that $f \circ h = u$, i.e. the following diagram commutes:



Proof: As injective and projective objects are dual to each other, it is enough to prove (a) as (b) follows from applying (a) in the dual category $\mathcal{A}^{\text{op}}$.

For (a), forward implication: Let I be injective in $\mathcal{A}$ and $f : A \to B$ be any monomorphism in $\mathcal{A}$. Due to (**2.2.4**(c)) we have $\text{Ker}(f) = 0$, then due to (**2.4.4**) we know that the short sequence $0 \to A \xrightarrow{f} B \to \text{Coker}(f) \to 0$ is exact.

Applying $\text{Hom}_{\mathcal{A}}(\_, I)$ which is exact, we have that $0 \to \text{Hom}_{\mathcal{A}}(\text{Coker}(f), I) \to \text{Hom}_{\mathcal{A}}(B, I) \xrightarrow{f^*} \text{Hom}_{\mathcal{A}}(A, I) \to 0$ is also exact due to (**2.4.8**). With (**2.2.4**(d)) we have that $f^* : \text{Hom}_{\mathcal{A}}(B, I) \to \text{Hom}_{\mathcal{A}}(A, I)$ is an epimorphism in $\text{SET}_{\mathcal{U}}$, which implies the surjectivity of f^* as a function. The claim then follows as for every $u \in \text{Hom}_{\mathcal{A}}(A, I)$ there exists a $i : B \to I$ with $f^*(i) = i \circ f = u$ as claimed.

For (a), backward implication: Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in $\mathcal{A}$. Since $\text{Hom}_{\mathcal{A}}(\_, I)$ is already left exact due to (**2.4.9**), we have that $0 \to \text{Hom}_{\mathcal{A}}(C, I) \xrightarrow{g^*} \text{Hom}_{\mathcal{A}}(B, I) \xrightarrow{f^*} \text{Hom}_{\mathcal{A}}(A, I)$ is also exact because of (**2.4.8**).

Due to (**2.2.4**(d)), it suffices to show that f^* an epimorphism, because $0 \to \text{Hom}_{\mathcal{A}}(C, I) \xrightarrow{g^*} \text{Hom}_{\mathcal{A}}(B, I) \xrightarrow{f^*} \text{Hom}_{\mathcal{A}}(A, I) \to 0$ would be exact and thus $\text{Hom}_{\mathcal{A}}(\_, I)$ would be exact due to (**2.4.8**). Since we assume the diagram condition for an object I in $\mathcal{A}$ and since $f : A \to B$ is a monomorphism in $\mathcal{A}$, we have that f^* is surjective and an epimorphism, and thus the claim follows. □

3.1.3 Lemma (Split Exact Sequences) (5.19, 5.20 [Bö20]**), (0136** [JC21]**), (II.2.4.5** [Mor20]**):** The following statements hold:

(a) I is injective if and only if every short exact sequence $0 \to I \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ splits, as defined in (**2.4.5**).

(b) If P is projective if and only if every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ in $\mathcal{A}$ splits, as defined in (**2.4.5**).

Proof: As (b) is equivalent to applying (a) in $\mathcal{A}^{\text{op}}$, it is enough to show that (a) is true.

For (a), forward implication: Let $0 \to I \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in $\mathcal{A}$. Using the notation and the equivalence from (**3.1.2**(a)), we choose $A = I$ and $u = 1_I$. This implies that there exists a $i : B \to I$, such that $i \circ f = u = 1_I$ follows. Hence, the short exact sequence splits.

For (a), backward implication: Let $f : A \to B$ be any monomorphism and let $u : A \to I$ be any morphism in $\mathcal{A}$. For the pushout $(I \sqcup_A B, \iota_I, \iota_B)$ in $\mathcal{A}$ induced from the morphisms f and and u, we have that ι_I is a monomorphism due to f being a monomorphism and (**2.6.3**(d)) rotated. Therefore, the bottom row of the diagram:

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & B & & & & \\ & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \iota_B} & & & & \\ 0 & \to & I & \xrightarrow{\iota_I} & I \sqcup_A B & \to & \mathrm{Coker}(\iota_I) & \to & 0 \end{array},$$

is exact and thus splits, giving us a morphism $h : I \sqcup_A B \to I$ such that $(h \circ \iota_B) \circ f = u$. Thus if we choose $i : B \to I$ as $i = h \circ \iota_B$, I fulfills the conditions (**3.1.2**(a)) and is thus injective. □

3.1.4 Example (Projective Objects): Let R be a $\mathcal{U}$-ring. R is then a projective object in ${}_R\textsc{Mod}_\mathcal{U}$ and $\textsc{Mod}_{R\mathcal{U}}$ since for every short exact sequence in these categories of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} R \to 0$, we can find an R-linear mapping $h : R \to B$, $1_R \mapsto a$, whereby $a \in g^{-1}(1_R)$ such that $g \circ h = \mathrm{id}_R$, (such an $a \in g^{-1}(1_R)$ exists since g is surjective). These short exact sequences split and thus due to (**3.1.3**(b)), R is a projective object.

3.1.5 Lemma (Products and Coproducts) (II.2.4.3 [Mor20]**):** Let $(A_i)_{i \in I}$ be a $\mathcal{U}$-collection of objects in $\mathcal{A}$, i.e we have $I \in \mathcal{U}$, then the following applies:

(a) If A_i is injective for all $i \in I$, then $\prod_{i \in I} A_i$ is injective, given it exists in $\mathcal{A}$.

(b) If A_i is projective for all $i \in I$, $\coprod_{i \in I} A_i$ is projective, given it exists in $\mathcal{A}$.

Proof: As (b) is equivalent to applying (a) in $\mathcal{A}^{\mathrm{op}}$, it is enough to show that (a) is true.

For (a): For an object A in $\mathcal{A}$, $\mathrm{Hom}_\mathcal{A}(\_, A) : \mathcal{A}^{\mathrm{op}} \to \textsc{Set}_\mathcal{U}$ commutes with $\mathcal{U}$-small indexed limits due to (**1.5.10**). This generalizes to $\mathrm{Hom}_\mathcal{A}(\_, \prod_{i \in I} A_i) : \mathcal{A}^{\mathrm{op}} \to \textsc{Set}_\mathcal{U}$ and $\prod_{i \in I} \mathrm{Hom}_\mathcal{A}(\_, A_i) : \mathcal{A}^{\mathrm{op}} \to \textsc{Set}_\mathcal{U}$ being naturally isomorphic. Due to (**1.5.17**(a)), i.e. due to the forgetful functor $\mathrm{For} : \textsc{Ab}_\mathcal{U} \to \textsc{Set}_\mathcal{U}$ commuting with $\mathcal{U}$-small limits, we have that $\mathrm{Hom}_\mathcal{A}(\_, \prod_{i \in I} A_i) : \mathcal{A}^{\mathrm{op}} \to \textsc{Ab}_\mathcal{U}$ and $\prod_{i \in I} \mathrm{Hom}_\mathcal{A}(\_, A_i) : \mathcal{A}^{\mathrm{op}} \to \textsc{Ab}_\mathcal{U}$ are also naturally isomorphic.

Since group-theoretic and category-theoretic kernels and cokernels coincide in $\textsc{Ab}_\mathcal{U}$, as seen in (**2.2.2**), it is clear that kernels and cokernels in $\textsc{Ab}_\mathcal{U}$ commute with $\mathcal{U}$-indexed products in $\textsc{Ab}_\mathcal{U}$, given they exist. Therefore, using (**2.4.8**) implies that $\mathrm{Hom}_\mathcal{A}(\_, \prod_{i \in I} A_i)$ is exact, since $\mathrm{Hom}_\mathcal{A}(\_, A_i)$ is exact for each $i \in I$. Thus $\prod_{i \in I} A_i$ must be injective. □

3.1.6 Definitions (Enough Projectives and Enough Injectives) (II.2.4.1 [Mor20]**), (II.14** [Mit65]**):**

(a) An abelian $\mathcal{U}$-category $\mathcal{A}$ has *enough injectives* if for every object A, there exists an injective object I in $\mathcal{A}$ and a monomorphism $A \to I$ in $\mathcal{A}$.

(b) An abelian $\mathcal{U}$-category $\mathcal{A}$ has *enough projectives* if for every object A, there exists a projective object P in $\mathcal{A}$ and an epimorphism $P \to A$ in $\mathcal{A}$.

3.1.7 Example (Enough Projectives and Enough Injectives): Later in (**3.4.8**), we will state and prove that for a $\mathcal{U}$-ring R, ${}_R\textsc{Mod}_\mathcal{U}$ and $\textsc{Mod}_{R\mathcal{U}}$ have enough injectives and enough projectives. This however will require generators and cogenerators as discussed in **Section 3.2**.

3.2 Generators and Cogenerators

Let $\mathcal{C}$ and $\mathcal{D}$ be a $\mathcal{U}$-categories and let $\mathcal{A}$ and $\mathcal{B}$ be abelian $\mathcal{U}$-categories.

3.2.1 Definitions (Generators and Cogenerators) (II.3.1.1 [Mor20]**):**

(a) G is a *generator of* $\mathcal{C}$ if $\mathrm{Hom}_\mathcal{C}(G, \_) : \mathcal{C} \to \textsc{Set}_\mathcal{U}$ is conservative, as defined in (**1.3.5**(d)).

(b) C is a *cogenerator of* $\mathcal{C}$ if $\mathrm{Hom}_\mathcal{C}(\_, C) : \mathcal{C}^{\mathrm{op}} \to \textsc{Set}_\mathcal{U}$ is conservative, as defined in (**1.3.5**(d)).

It is clear that generators and cogenerators are dual to each other, i.e. A is a generator in $\mathcal{C}$ if and only if it is a cogenerator in $\mathcal{C}^{\text{op}}$.

3.2.2 Example (Generators): A singleton $\{\star\}$ is a generator in $\text{SET}_{\mathcal{U}}$, since for a function $f : A \to B$ in $\text{SET}_{\mathcal{U}}$ whereby the induced mapping $f_* : \text{Hom}_{\text{SET}_{\mathcal{U}}}(\{\star\}, A) \to \text{Hom}_{\text{SET}_{\mathcal{U}}}(\{\star\}, B)$ is an isomorphism, so is f an isomorphism.

3.2.3 Lemma (Faithful Functors Reflect Monomorphisms and Epimorphisms) (II.7.1 [Mit65]**):** Let $F : \mathcal{C} \to \mathcal{D}$ be a faithful functor. Then F *reflects monomorphisms* (and *reflects epimorphisms*), i.e. for every morphism $f : A \to B$ such that $Ff : FA \to FB$ is a monomorphism (respectively epimorphism), we have that $f : A \to B$ is a monomorphism (respectively epimorphism).

Proof: For F reflecting monomorphisms: Let $f : A \to B$ be a morphism in $\mathcal{C}$ such that Ff is a monomorphism in $\mathcal{D}$. As Ff is a monomorphism, the left composition $(Ff)_* : \text{Hom}_{\mathcal{D}}(FW, FA) \to \text{Hom}_{\mathcal{D}}(FW, FB)$, $g \mapsto Ff \circ g$ is injective for all objects W in $\mathcal{C}$. With $(Ff)_*$ restricted to $Fg \mapsto F(f \circ g)$ for all morphisms $g : W \to A$ in $\mathcal{C}$ being still injective, we see that $f_* : \text{Hom}_{\mathcal{D}}(W, A) \to \text{Hom}_{\mathcal{D}}(W, B)$, $g \mapsto f \circ g$ must also be injective as F is faithful. This implies that f is a monomorphism.

For F reflecting epimorphisms: Let $f : A \to B$ be a morphism in $\mathcal{C}$ such that Ff is an epimorphism in $\mathcal{D}$. As Ff is an epimorphism, the right composition $(Ff)^* : \text{Hom}_{\mathcal{D}}(FB, FW) \to \text{Hom}_{\mathcal{D}}(FA, FW)$, $h \mapsto h \circ Ff$ is injective for all objects W in $\mathcal{C}$. With $(Ff)^*$ restricted to $Fh \mapsto F(h \circ f)$ for all morphisms $h : B \to W$ in $\mathcal{C}$ being still injective, we see that $f^* : \text{Hom}_{\mathcal{D}}(B, W) \to \text{Hom}_{\mathcal{D}}(A, W)$, $h \mapsto h \circ f$ is injective as F is faithful. This implies that f is an epimorphism. $\square$

3.2.4 Lemma (Faithful Conservative Equivalences) (Ex 4.12 [Sch21]**):**

(a) Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor, then F is conservative if and only if it is faithful.

(b) Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor, then F is conservative if and only if it is faithful.

Proof: Due to duality, it is enough to prove (a), since (b) derives from (a) applied to the opposite functor $F^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathcal{B}^{\text{op}}$ due to (**2.4.7**). This is because the property of functors being conservative or faithful is preserved between dualities.

For (a), forward implication: Let A and B by any two objects in $\mathcal{A}$. As F is additive, we want to show that the F-induced group homomorphism $F_{A,B} : \text{Hom}_{\mathcal{A}}(A, B) \to \text{Hom}_{\mathcal{A}}(FA, FB), f \mapsto Ff$ is injective by showing that is has a trivial kernel i.e. $Ff = 0_{FA,FB}$ implies $f = 0_{A,B}$.

Let $\iota : \text{Ker}(f) \to A$ be the kernel morphism in $\mathcal{A}$. Then $F\iota : F(\text{Ker}(f)) \cong \text{Ker}(Ff) \to FA$ is the kernel morphism of Ff in $\mathcal{B}$ due to (**2.4.8**). Due to the properties of zero morphisms, $Ff = 0_{FA,FB} : FA \to FB$ has the isomorphism $F\iota = 1_{FA} : \text{Ker}(Ff) = FA \to FA$ as the kernel morphism. As F is conservative, this implies that $\iota : \text{Ker}(f) \to A$ is an isomorphism, i.e. $\text{Ker}(f) \cong A$, which implies that $f = 0_{A,B}$ is the zero morphism due to the characterizations of zero morphisms.

For (a), backward implication: As F is faithful, we have that F reflects monomorphisms and epimorphisms due to (**3.2.3**). For all morphisms $f : A \to B$ in $\mathcal{A}$ such that $Ff : FA \to FB$ is an isomorphism, we then have that f is a monomorphism and epimorphism due to (**3.2.3**). Then since $\mathcal{A}$ is an abelian $\mathcal{U}$-category, we have that f is an isomorphism due to (**2.3.4**). $\square$

3.2.5 Lemma (Generators and Cogenerators) (II.3.1.3 [Mor20]**), (Ex 4.13** [Sch21]**):** The following statements are true:

(a) If $\mathcal{A}$ has a generator G, then for a morphism $f : A \to B$ in $\mathcal{A}$, whereby the left composition $f_* : \text{Hom}_{\mathcal{A}}(G, A) \to \text{Hom}_{\mathcal{A}}(G, B)$ is surjective, f is an epimorphism.

(b) If $\mathcal{A}$ has a cogenerator C, then for a morphism $f : A \to B$ in $\mathcal{A}$, whereby the right composition $f^* : \mathrm{Hom}_{\mathcal{A}}(B, C) \to \mathrm{Hom}_{\mathcal{A}}(A, C)$ is surjective, f is a monomorphism.

Proof: Due to duality, it is again enough to prove (a).

For (a): Since $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ is conservative and is left exact due to (**2.4.9**), we apply (**3.2.4**(a)) which implies that $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ is faithful and thus reflects monomorphisms and epimorphisms due to (**3.2.3**). Since $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ maps f to f_*, and f_* is surjective, i.e. an epimorphism in $\mathrm{Ab}_{\mathcal{U}}$, f must then be an epimorphism. □

3.2.6 Lemma (Subobjects and Quotients are $\mathcal{U}$-Small) (II.3.1.3 [Mor20]**):** If $\mathcal{A}$ has a generator G or a cogenerator C, then for any object A in $\mathcal{A}$ the sets $\mathrm{Sub}(A)$ and $\mathrm{Quot}(A)$, as defined in (**2.5.1**), are $\mathcal{U}$-small sets.

Proof: As subobjects and quotients are in bijection to each other due to (**2.5.2**), it is enough to show that $\mathrm{Sub}(A)$ is $\mathcal{U}$-small.

If $\mathcal{A}$ has a generator G: For any subobject (B, ι) of A, we claim that (B, ι) injects to the image of $\iota_* : \mathrm{Hom}_{\mathcal{A}}(G, B) \to \mathrm{Hom}_{\mathcal{A}}(G, A)$, i.e. (B, ι) injects to $\mathrm{Im}(\iota_*) \in \mathcal{P}(\mathrm{Hom}_{\mathcal{A}}(G, A))$. To show this, let (B_1, ι_1) and (B_2, ι_2) be two subobjects of A such that $\mathrm{Im}(\iota_{1*}) = \mathrm{Im}(\iota_{2*})$. Since ι_1 and ι_2 are monomorphisms, ι_{1*} and ι_{2*} are injective and thus $\mathrm{Hom}_{\mathcal{A}}(G, B_1) \cong \mathrm{Im}(\iota_{1*}) = \mathrm{Im}(\iota_{1*}) \cong \mathrm{Hom}_{\mathcal{A}}(G, B_2)$.

Since $\mathcal{A}$ is abelian, let $(B_1 \times_A B_2, p_{B_1}, p_{B_2})$ be the pullback induced from the morphisms ι_1 and ι_2. Due to $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ being left exact as seen in (**2.4.9**), we have the isomorphisms in $\mathrm{Ab}_{\mathcal{U}}$: $\mathrm{Hom}_{\mathcal{A}}(G, B_1 \times_A B_2) \cong \mathrm{Hom}_{\mathcal{A}}(G, B_1) \times_{\mathrm{Hom}_{\mathcal{A}}(G,A)} \mathrm{Hom}_{\mathcal{A}}(G, B_2) \cong \mathrm{Hom}_{\mathcal{A}}(G, B_1) \times_{\mathrm{Hom}_{\mathcal{A}}(G,A)} \mathrm{Hom}_{\mathcal{A}}(G, B_1)$, with the final object being the fiber product induced by ι_{1*} twice, and with final isomorphism due to $\mathrm{Hom}_{\mathcal{A}}(G, B_1) \cong \mathrm{Hom}_{\mathcal{A}}(G, B_2)$. Furthermore, since $\mathrm{Hom}_{\mathcal{A}}(G, B_1) \times_{\mathrm{Hom}_{\mathcal{A}}(G,A)} \mathrm{Hom}_{\mathcal{A}}(G, B_1)$ is isomorphic to $\mathrm{Hom}_{\mathcal{A}}(G, B_1)$ due to ι_{1*} being injective, we have $\mathrm{Hom}_{\mathcal{A}}(G, B_1 \times_A B_2) \cong \mathrm{Hom}_{\mathcal{A}}(G, B_1) \cong \mathrm{Hom}_{\mathcal{A}}(G, B_2)$ induced by the mappings $p_{B_1 *}$ and $p_{B_2 *}$.

Therefore, as $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ is conservative since G is a generator, we have that p_{B_1} and p_{B_2} are isomorphisms, and therefore $B_1 \cong B_2$. Implying that (B_1, ι_1) and (B_2, ι_2) are the same objects.

Therefore, we have an injection of subobjects of A to elements of $\mathcal{P}(\mathrm{Hom}_{\mathcal{A}}(G, A))$, which is a $\mathcal{U}$-set. We thus have that $\mathrm{Sub}(A)$ bijects to a $\mathcal{U}$-subset of $\mathcal{P}(\mathrm{Hom}_{\mathcal{A}}(G, A))$, and thus $\mathrm{Sub}(A)$ is $\mathcal{U}$-small.

If $\mathcal{A}$ has a cogenerator C: This follows directly from applying the above argument to $\mathcal{A}^{\mathrm{op}}$ which would now have a generator C. Due to duality, $\mathrm{Sub}(A)$ in $\mathcal{A}$ would be in bijection to $\mathrm{Quot}(A)$ in $\mathcal{A}^{\mathrm{op}}$, which is $\mathcal{U}$-small due to the above argument. □

3.2.7 Lemma (Generators and Cogenerators) (II.3.1.4 [Mor20]**), (4.4.2** [Sch21]**):** Let $\mathcal{A}$ be an abelian $\mathcal{U}$-category that contains all $\mathcal{U}$-small indexed coproducts, then for an object G in $\mathcal{A}$, the following are equivalent:

(i) G is a generator of $\mathcal{A}$.

(ii) For all objects A of $\mathcal{A}$, there exists an $I \in \mathcal{U}$, such that there exists an epimorphism $\alpha_A : \coprod_{i \in I} G \to A$.

(iii) $\mathrm{Hom}_{\mathcal{A}}(G, \_) : \mathcal{A} \to \mathrm{Ab}_{\mathcal{U}}$ is a faithful functor.

Analogously, if $\mathcal{A}$ instead is an abelian $\mathcal{U}$-category that contains all $\mathcal{U}$-small indexed products, then for an object C in $\mathcal{A}$, the following dual statements are also equivalent:

(iv) C is a cogenerator of $\mathcal{A}$.

(v) For all objects A of $\mathcal{A}$, there exists an $I \in \mathcal{U}$, such that there exists a monomorphism $\alpha_A : A \to \prod_{i \in I} C$.

(vi) $\mathrm{Hom}_{\mathcal{A}}(\_, C) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}_{\mathcal{U}}$ is a faithful functor.

Proof: Due to duality, it is enough to prove the first three equivalences, as the last three equivalences can be derived by applying the first three equivalences on $\mathcal{A}^{\mathrm{op}}$.

For (i) implies (ii): Let A be any object in $\mathcal{A}$. There exists a bijection of $\mathcal{U}$-sets:

$$\varphi : \mathrm{Hom}_{\mathrm{SET}_{\mathcal{U}}}(\mathrm{Hom}_{\mathcal{A}}(G,A), \mathrm{Hom}_{\mathcal{A}}(G,A)) \to \mathrm{Hom}_{\mathcal{A}}\left(\coprod_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)} G, A\right),$$

given by the function $f : \mathrm{Hom}_{\mathcal{A}}(G,A) \to \mathrm{Hom}_{\mathcal{A}}(G,A)$ being mapped to $\varphi(f) : \coprod_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)} G \to A$, which is determined by the morphisms $(f(i) : G \to A)_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)}$ and the universal property of $\coprod_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)} G$. φ has the inverse mapping ψ which maps $g : \coprod_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)} G \to A$, determined and induced uniquely by the morphisms $(g_i : G \to A)_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)}$, to $\psi(g) : \mathrm{Hom}_{\mathcal{A}}(G,A) \to \mathrm{Hom}_{\mathcal{A}}(G,A)$, which maps $i \mapsto (g_i : G \to A)$.

The identity $\mathrm{id}_{\mathrm{Hom}_{\mathcal{A}}(G,A)} : \mathrm{Hom}_{\mathcal{A}}(G,A) \to \mathrm{Hom}(G,A)$ maps to $\varphi(\mathrm{id}_{\mathrm{Hom}_{\mathcal{A}}(G,A)}) : \coprod_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)} G \to A$. We claim that the left composition $\varphi(\mathrm{id}_{\mathrm{Hom}_{\mathcal{A}}(G,A)})_* : \mathrm{Hom}_{\mathcal{A}}(G, \coprod_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)} G) \to \mathrm{Hom}_{\mathcal{A}}(G,A)$, $g \mapsto \varphi(\mathrm{id}_{\mathrm{Hom}_{\mathcal{A}}(G,A)}) \circ g$ is surjective. Let $h \in \mathrm{Hom}_{\mathcal{A}}(G,A)$, then since $\varphi(\mathrm{id}_{\mathrm{Hom}_{\mathcal{A}}(G,A)})$ is the morphism induced from the collection of morphisms $(i)_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)}$, define $\iota_h \in \mathrm{Hom}_{\mathcal{A}}(G, \coprod_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)} G)$ as the inclusion to $\coprod_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)} G$ on the h-component. Then we have $\varphi(\mathrm{id}_{\mathrm{Hom}_{\mathcal{A}}(G,A)}) \circ \iota_h = h$. Thus $\varphi(\mathrm{id}_{\mathrm{Hom}_{\mathcal{A}}(G,A)})_*$ is surjective.

Using (**3.2.5**(a)) gives us that $\alpha_A = \varphi(\mathrm{id}_{\mathrm{Hom}_{\mathcal{A}}(G,A)}) : \coprod_{i\in\mathrm{Hom}_{\mathcal{A}}(G,A)} G \to A$ is an epimorphism, whereby $I = \mathrm{Hom}_{\mathcal{A}}(G,A) \in \mathcal{U}$.

For (ii) implies (iii): For two objects A and B in $\mathcal{A}$, we want to show that the group homomorphism induced from $\mathrm{Hom}_{\mathcal{A}}(G,\_)$, i.e. $\mathrm{Hom}_{\mathcal{A}}(G,\_)_{A,B} : \mathrm{Hom}_{\mathcal{A}}(A,B) \to \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{A}}(G,A), \mathrm{Hom}_{\mathcal{A}}(G,B)), f \mapsto f_*$, is injective by showing that it has a trivial kernel i.e. $f_* = 0_{\mathrm{Hom}_{\mathcal{A}}(G,A),\mathrm{Hom}_{\mathcal{A}}(G,B)}$ implies $f = 0_{A,B}$.

Let $f : A \to B$ be a morphism in $\mathcal{A}$ such that $f_* : \mathrm{Hom}_{\mathcal{A}}(G,A) \to \mathrm{Hom}_{\mathcal{A}}(G,B)$ is the zero morphism. There exists an index set $I \in \mathcal{U}$ and an epimorphism $\alpha_A : \coprod_{i\in I} G \to A$ which is uniquely induced by the morphisms $(\alpha_{Ai} : G \to A)_{i\in I}$. For all $i \in I$ we then have $f \circ \alpha_{Ai} = 0_{A,B} \circ \alpha_{Ai} = 0_{G,B}$. This implies $f \circ \alpha_A = 0_{A,B} \circ \alpha_A$ and since α_A is an epimorphism (i.e. right composition α_A^* is injective), we have that $f = 0_{A,B}$. Thus $\mathrm{Hom}_{\mathcal{A}}(G,\_)$ is faithful.

For (iii) implies (i): Since $\mathrm{Hom}_{\mathcal{A}}(G,\_)$ is left exact due to (**2.4.9**), the claim follows due to (**3.2.4**(a)). ☐

3.2.8 Examples (Generators) (II.3.1.2(2) [Mor20]**):** Now that we have equivalent characterizations of generators and cogenerators in abelian $\mathcal{U}$-categories, we can immediately find some examples:

(a) Let R be a $\mathcal{U}$-ring, then R is a generator in ${}_R\mathrm{MOD}_{\mathcal{U}}$ as a $\mathcal{U}$-left-R-module. For every $\mathcal{U}$-left-R-module M, we have a morphism $\alpha_A : \bigoplus_{i\in\mathrm{Hom}_{{}_R\mathrm{MOD}_{\mathcal{U}}}(R,M)} R \to M$ induced by the R-linear mappings $(i : R \to M)_{i\in\mathrm{Hom}_{{}_R\mathrm{MOD}_{\mathcal{U}}}(R,M)}$. α_A is an epimorphism as for all objects $m \in M$, there exists an R-linear mapping $i : R \to M$ such that $i(1_R) = m$ and therefore $\alpha_A(1_{Ri}) = i(1_R) = m$, for 1_{Ri} being the multiplicative unit on the i-component. Thus, R fulfills condition (**3.2.7**(ii)).

Analogously due to duality and (**1.3.9**), we have that R is a generator of $\mathrm{MOD}_{R\mathcal{U}}$.

(b) $\mathbb{Z}$ is a generator in $\mathrm{AB}_{\mathcal{U}}$ with the help of (**3.2.8**(a)), since every abelian $\mathcal{U}$-group has the same structure as a $\mathcal{U}$-left-$\mathbb{Z}$-module.

3.2.9 Lemma (Projective Generators and Injective Cogenerators) (II.3.1.5 [Mor20]**):** Let $\mathcal{A}$ be an abelian $\mathcal{U}$-category that contains all $\mathcal{U}$-small indexed coproducts, then for an object G in $\mathcal{A}$, the following are equivalent:

(i) G is a projective generator of $\mathcal{A}$.

(ii) $\mathrm{Hom}_{\mathcal{A}}(G,\_) : \mathcal{A} \to \mathrm{AB}_{\mathcal{U}}$ is a faithful and exact functor.

(iii) G is projective and for every nonzero object A in $\mathcal{A}$, there exists a nonzero morphism $\alpha_A : G \to A$.

Analogously, if $\mathcal{A}$ instead is an abelian $\mathcal{U}$-category that contains all $\mathcal{U}$-small indexed products, then for an object C in $\mathcal{A}$, the following dual statements are also equivalent:

(iv) C is an injective cogenerator of $\mathcal{A}$.

(v) $\mathrm{Hom}_{\mathcal{A}}(\_, C) : \mathcal{A}^{\mathrm{op}} \to \mathrm{AB}_{\mathcal{U}}$ is a faithful and exact functor.

(vi) C is injective and for every nonzero object A in $\mathcal{A}$, there exists a nonzero morphism $\alpha_A : A \to C$.

Proof: Due to duality, it is enough to prove the first three equivalences, as the last three equivalences can be derived by applying the first three equivalences on $\mathcal{A}^{\mathrm{op}}$.

For (i) equivalent to (ii): G being projective is per definition equivalent to $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ being exact. G being a generator is equivalent to $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ being faithful due to (**3.2.7**). In summary, G being a projective generator is equivalent to $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ being faithful and exact.

For (i) implies (iii): We already have that G is projective. Let A be a nonzero object in $\mathcal{A}$, then $1_A \neq 0_{A,A}$ due to (**2.1.3**). Thus we have that $(1_A)_* = \mathrm{id}_{\mathrm{Hom}_{\mathcal{A}}(G,A)} : \mathrm{Hom}_{\mathcal{A}}(G, A) \to \mathrm{Hom}_{\mathcal{A}}(G, A)$ is not the zero morphism as $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ is faithful. This implies that $\mathrm{Hom}_{\mathcal{A}}(G, A)$ is a nonzero object of $\mathrm{AB}_{\mathcal{U}}$ due to (**2.1.3**). Thus, there exists a nonzero morphism $\alpha_A \in \mathrm{Hom}_{\mathcal{A}}(G, A)$.

For (iii) implies (ii): We already have that $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ is exact as G is projective. Let $f : A \to B$ be a nonzero morphism in $\mathcal{A}$, i.e. $f \neq 0_{A,B}$, then we want to show that $f_* : \mathrm{Hom}_{\mathcal{A}}(G, A) \to \mathrm{Hom}_{\mathcal{A}}(G, B)$ is a nonzero morphism.

Since f is a nonzero morphism, $\mathrm{Im}(f)$ is a nonzero object (as the cokernel would otherwise be B). Thus due to (iii), there exists a nonzero morphism $\alpha_{\mathrm{Im}(f)} : G \to \mathrm{Im}(f)$. Using the decomposition in (**2.2.5**), we split $f = \iota \circ p$, whereby $p = u_f \circ t_f : A \to \mathrm{Im}(f)$ which is an epimorphism (as a composition of two epimorphisms), and $\iota = v_f : \mathrm{Im}(f) \to B$ which is a monomorphism. Since G is projective and p is an epimorphism, we can use (**3.1.2**(b)) on G, p and $\alpha_{\mathrm{Im}(f)}$, which implies that there exists a morphism $h : G \to A$ in $\mathcal{A}$ such that $p \circ h = \alpha_{\mathrm{Im}(f)}$. Then we have:

$$f_*(h) = f \circ h = \iota \circ p \circ h = \iota \circ \alpha_{\mathrm{Im}(f)}.$$

Since $\alpha_{\mathrm{Im}(f)}$ is nonzero, we have $\mathrm{Ker}(\alpha_{\mathrm{Im}(f)}) \ncong G$ and since ι is a monomorphism, through universal properties it follows that $\mathrm{Ker}(\iota \circ \alpha_{\mathrm{Im}(f)}) \cong \mathrm{Ker}(\alpha_{\mathrm{Im}(f)}) \ncong G$ and thus $f_*(h) = \iota \circ \alpha_{\mathrm{Im}(f)}$ is nonzero. As f_* is nonzero, the mapping $f \mapsto f_*$ is injective and thus $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ is faithful. □

3.2.10 Example (Projective Generators) (II.3.1.2(4) [Mor20]**):** Let $\mathcal{C}$ be a $\mathcal{U}$-small category and R be a $\mathcal{U}$-ring. Then since ${}_R\mathrm{MOD}_{\mathcal{U}}$ is a $\mathcal{U}$-category, we have that $\mathrm{PSH}(\mathcal{C}, R)$ is a $\mathcal{U}$-category due to (**1.3.4**(b)). Furthermore, since ${}_R\mathrm{MOD}_{\mathcal{U}}$ contains $\mathcal{U}$-small colimits due to (**1.5.12**(b)), $\mathrm{PSH}(\mathcal{C}, R)$ also contains $\mathcal{U}$-small colimits. As ${}_R\mathrm{MOD}_{\mathcal{U}}$ is an abelian $\mathcal{U}$-category, we have that $\mathrm{PSH}(\mathcal{C}, R)$ is an abelian $\mathcal{U}$-category due to (**2.3.2**(c)).

Let C be an object in $\mathcal{C}$. We then define the *presheaf of $\mathcal{U}$-left-R-modules on C*, denoted by $R^{(C)}$, as the presheaf $\langle\_\rangle \circ \mathrm{Hom}_{\mathcal{C}}(\_, C)$ in $\mathrm{PSH}(\mathcal{C}, R)$, where $\langle\_\rangle$ is the free functor from (**1.3.7**(a)). Therefore there exists a presheaf $\coprod_{C \in \mathrm{Ob}(\mathcal{C})} R^{(C)}$ in $\mathrm{PSH}(\mathcal{C}, R)$ that maps every object A in $\mathcal{C}$ to the $\mathcal{U}$-left-R-module $\bigoplus_{C \in \mathrm{Ob}(\mathcal{C})} \langle \mathrm{Hom}_{\mathcal{C}}(A, C)\rangle$. We claim that $\coprod_{C \in \mathrm{Ob}(\mathcal{C})} R^{(C)}$ is a projective generator in $\mathrm{PSH}(\mathcal{C}, R)$.

Explanation: Let F be any presheaf in $\mathrm{PSH}(\mathcal{C}, R)$. Due to (**1.5.10**) and the forgetful functor $\mathrm{For} : \mathrm{AB}_{\mathcal{U}} \to \mathrm{SET}_{\mathcal{U}}$ commuting with $\mathcal{U}$-small limits due to (**1.5.17**(a)), we have a group isomorphism $\mathrm{Hom}_{\mathrm{PSH}(\mathcal{C},R)}(\coprod_{C \in \mathrm{Ob}(\mathcal{C})} R^{(C)}, F) \cong \prod_{C \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathrm{PSH}(\mathcal{C},R)}(R^{(C)}, F)$.

For the forgetful functor $\mathrm{For} : {}_R\mathrm{MOD}_{\mathcal{U}} \to \mathrm{SET}_{\mathcal{U}}$, we also have the adjunction $(\langle\_\rangle, \mathrm{For})$ due to (**1.3.7**(a)), and therefore since $R^{(C)} = \langle\_\rangle \circ h_C$, we then have the bijection

$\prod_{C\in\mathrm{Ob}(\mathcal{C})}\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},R)}(R^{(C)},F)\cong\prod_{C\in\mathrm{Ob}(\mathcal{C})}\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},\mathrm{Set}_{\mathcal{U}})}(h_C,\mathrm{For}F)$ as sets. With the help of the *Yoneda lemma* from (**1.4.5**), we then have the bijection $\prod_{C\in\mathrm{Ob}(\mathcal{C})}\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},\mathrm{Set}_{\mathcal{U}})}(h_C,\mathrm{For}F)\cong\prod_{C\in\mathrm{Ob}(\mathcal{C})}FC$ as sets. By explicit calculation, it can be shown that composing the bijections above gives us a group isomorphism $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},R)}(\coprod_{C\in\mathrm{Ob}(\mathcal{C})}R^{(C)},F)\cong\prod_{C\in\mathrm{Ob}(\mathcal{C})}FC$.

Due to the functoriality of limits and colimits and due to the construction of the previous group isomorphism $\prod_{C\in\mathrm{Ob}(\mathcal{C})}\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},R)}(R^{(C)},F)\cong\prod_{C\in\mathrm{Ob}(\mathcal{C})}FC$, we have that this isomorphism extends into a natural transformation of functors from $\mathrm{PSh}(\mathcal{C},R)$ to $\mathrm{Ab}_{\mathcal{U}}$:

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},R)}\left(\coprod_{C\in\mathrm{Ob}(\mathcal{C})}R^{(C)},\_\right)\cong\prod_{C\in\mathrm{Ob}(\mathcal{C})}(\_)C.$$

Since $\prod_{C\in\mathrm{Ob}(\mathcal{C})}(\_)C$ is clearly a faithful functor due to how natural transformations between presheaves are constructed, we see that $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},R)}(\coprod_{C\in\mathrm{Ob}(\mathcal{C})}R^{(C)},\_)$ is faithful. We also claim that $\prod_{C\in\mathrm{Ob}(\mathcal{C})}(\_)C$ is an exact functor: For a natural transformation $u:F\to G$ in $\mathrm{PSh}(\mathcal{C},R)$, we have that the kernel and cokernel of u are determined object-wise, as seen in (**2.3.2**(c)). Therefore for any object C in $\mathcal{C}$, we have the corresponding kernel and cokernel morphisms $\mathrm{Ker}(uC)\to FC$ and $GC\to\mathrm{Coker}(uC)$. These morphisms altogether commute with the product in $\prod_{C\in\mathrm{Ob}(\mathcal{C})}(\_)C$ due to the fact that group-theoretic kernels and cokernels coincide with category-theoretic kernels and cokernels, as seen in (**2.2.2**). The exactness of $\prod_{C\in\mathrm{Ob}(\mathcal{C})}(\_)C$ then follows from (**2.4.8**), which also implies the exactness of $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},R)}(\coprod_{C\in\mathrm{Ob}(\mathcal{C})}R^{(C)},\_)$.

Since $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},R)}(\coprod_{C\in\mathrm{Ob}(\mathcal{C})}R^{(C)},\_)$ is a faithful and exact functor, $\coprod_{C\in\mathrm{Ob}(\mathcal{C})}R^{(C)}$ is a projective generator in $\mathrm{PSh}(\mathcal{C},R)$ as claimed due to (**3.2.9**).

3.3 Embedding Theorems

Let $\mathcal{A}$ be an abelian $\mathcal{U}$-category.

3.3.1 Motivation (Embedding Theorems): In order to embed $\mathcal{U}$-small subcategories of $\mathcal{A}$ into a category of modules, which is the goal of *Mitchell's embedding theorem*, a sensible approach would be to choose an object G such that the additive left exact functor $\mathrm{Hom}_{\mathcal{A}}(G,\_):\mathcal{A}\to\mathrm{Ab}_{\mathcal{U}}$ fulfills certain properties e.g. faithfulness or exactness. Then we would restrict $\mathrm{Hom}_{\mathcal{A}}(G,\_)$ onto subcategories of $\mathcal{A}$.

After fixing an object G, we define $R=\mathrm{Hom}_{\mathcal{A}}(G,G)$ to be the endomorphisms on G, which defines a $\mathcal{U}$-ring with respect to the usual addition and multiplication given by composition. We can define for all objects W in $\mathcal{A}$ a $\mathcal{U}$-right-R-module structure on $\mathrm{Hom}_{\mathcal{A}}(G,W)$ with the multiplication $\mathrm{Hom}_{\mathcal{A}}(G,W)\times R\to\mathrm{Hom}_{\mathcal{A}}(G,W)$ given by $f\cdot r=f\circ r$. Due to the bilinearity of composition as well as its associativity, the $\mathcal{U}$-right-R-module structure is obvious. Since it is clear that for a morphism $f:A\to B$ in $\mathcal{A}$, we have an R-linear mapping $f_*:\mathrm{Hom}_{\mathcal{A}}(G,A)\to\mathrm{Hom}_{\mathcal{A}}(G,B)$, we have a functor $\mathrm{Hom}^R_{\mathcal{A}}(G,\_):\mathcal{A}\to\mathrm{Mod}_{R\mathcal{U}}$ (with R made explicit in our notation).

3.3.2 Lemma (Exact Inclusions) (A.4.4(d)(i) [Mor20]**):** Let $\mathcal{B}$ be a full $\mathcal{U}$-small subcategory of $\mathcal{A}$ with the following property: For all finite diagrams $D:\mathcal{I}\to\mathcal{A}$ such that $Di\in\mathrm{Ob}(\mathcal{B})$ for all $i\in\mathrm{Ob}(\mathcal{I})$, all possible limits and colimit objects and morphisms $\lim_{\mathcal{I}}D$, $\mathrm{colim}_{\mathcal{I}}D$ are in $\mathcal{B}$ (we denote this as a property of $\mathcal{B}$, that $\mathcal{B}$ is *stable under finite limits and colimits of* $\mathcal{A}$). We then have that $\mathcal{B}$ is an abelian $\mathcal{U}$-category and that the inclusion functor $\iota:\mathcal{B}\to\mathcal{A}$ is exact.

Proof: For $\mathcal{B}$ being an abelian $\mathcal{U}$-category: $\mathcal{B}$ is pre-additive, because for all objects B and C in $\mathcal{B}$ we have $\mathrm{Hom}_{\mathcal{B}}(B,C)=\mathrm{Hom}_{\mathcal{A}}(B,C)$ since $\mathcal{B}$ is a full $\mathcal{U}$-subcategory of $\mathcal{A}$. $\mathrm{Hom}_{\mathcal{B}}(B,C)$ retains the abelian $\mathcal{U}$-group structure of $\mathrm{Hom}_{\mathcal{A}}(B,C)$ and analogously the composition $\circ$ is still bilinear.

Since all finite limits and colimits exist in $\mathcal{A}$ due to (**2.3.5**), finite biproducts, kernels and cokernels must all exist in $\mathcal{B}$, as limits and colimits of finite diagrams as seen in **Section 1.5** and **Section 2.2**. Thus due to (**2.2.5**), every morphism $f:B\to C$ in $\mathcal{B}$ has the decomposition $A\xrightarrow{t_f}\mathrm{Coim}(f)\xrightarrow{u_f}\mathrm{Im}(f)\xrightarrow{v_f}B$ in $\mathcal{A}$ such that u_f is an isomorphism. Due to the properties of

$\mathcal{B}$, the same decomposition exists in $\mathcal{B}$ such that u_f is an isomorphism in $\mathcal{B}$. This makes $\mathcal{B}$ an abelian $\mathcal{U}$-category.

For ι being exact: The inclusion functor ι is clearly an additive functor and is exact due to kernels and cokernels in $\mathcal{B}$ and $\mathcal{A}$ coinciding with each other and (**2.4.8**). □

3.3.3 Lemma (Embedding Theorems) (A.4.4(d)(ii),(iii) [Mor20]**):** Let $\mathcal{B}$ be a full $\mathcal{U}$-small subcategory of $\mathcal{A}$ that is stable under finite limits and colimits of $\mathcal{A}$ and let G be a generator of $\mathcal{A}$ (i.e. such a generator G exists in $\mathcal{A}$). For every object A in $\mathcal{A}$, we have an epimorphism $\alpha_A : \coprod_{i \in \mathrm{Hom}_{\mathcal{A}}(G,A)} G \to A$ induced by the components $(i)_{i \in \mathrm{Hom}_{\mathcal{A}}(G,A)}$ due to (**3.2.7**).

We define $H = \coprod_{B \in \mathrm{Ob}(\mathcal{B})} \coprod_{i \in \mathrm{Hom}_{\mathcal{A}}(G,B)} G$ and define the morphism $\beta_B : H \to B$ for every object B in $\mathcal{B}$ as induced by α_B on the B-component and zero morphisms for every other component $C \in \mathrm{Ob}(\mathcal{B})$, $C \neq B$. Then let $S = \mathrm{Hom}_{\mathcal{A}}(H, H)$ be the $\mathcal{U}$-ring of endomorphisms, as defined in (**3.3.1**).

(a) $\mathrm{Hom}^S_{\mathcal{A}}(H, \_) : \mathcal{A} \to \mathrm{MOD}_{S\mathcal{U}}$ is faithful and if G is projective, then $\mathrm{Hom}^S_{\mathcal{A}}(H, \_)$ is exact.

(b) If G is projective, $\mathrm{Hom}^S_{\mathcal{A}}(H, \_)|_{\mathcal{B}} : \mathcal{B} \to \mathrm{MOD}_{S\mathcal{U}}$, which is the restriction of $\mathrm{Hom}^S_{\mathcal{A}}(H, \_) : \mathcal{A} \to \mathrm{MOD}_{S\mathcal{U}}$ onto $\mathcal{B}$, is fully faithful and exact.

Proof: For (a): Due to the universal property of coproducts, we have that every morphism $f : H \to W$ in $\mathcal{A}$ is uniquely induced by a collection of morphisms which are elements of $\prod_{B \in \mathrm{Ob}(\mathcal{B})} \prod_{i \in \mathrm{Hom}_{\mathcal{A}}(G,B)} \mathrm{Hom}_{\mathcal{A}}(G, W)$. This induces a natural isomorphism between $\mathrm{Hom}_{\mathcal{A}}(H, \_)$ and $\prod_{B \in \mathrm{Ob}(\mathcal{B})} \prod_{i \in \mathrm{Hom}_{\mathcal{A}}(G,B)} \mathrm{Hom}_{\mathcal{A}}(G, \_)$. Since G is a generator, $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ is faithful due to (**3.2.4**(a)), as a consequence $\prod_{B \in \mathrm{Ob}(\mathcal{B})} \prod_{i \in \mathrm{Hom}_{\mathcal{A}}(G,B)} \mathrm{Hom}_{\mathcal{A}}(G, \_)$ is also faithful and $\mathrm{Hom}_{\mathcal{A}}(H, \_)$ is faithful. It follows directly that $\mathrm{Hom}^S_{\mathcal{A}}(H, \_)$ is faithful.

If G is projective, then H is projective due to (**3.1.5**) and therefore $\mathrm{Hom}_{\mathcal{A}}(H, \_) : \mathcal{A} \to \mathrm{AB}_{\mathcal{U}}$ is exact. It follows directly that $\mathrm{Hom}^S_{\mathcal{A}}(H, \_)$ is also exact.

For (b): Since $\mathrm{Hom}^S_{\mathcal{A}}(H, \_)$ is faithful and exact due to (a) and since $\mathrm{Hom}^S_{\mathcal{A}}(H, \_)|_{\mathcal{B}}$ is a restriction of $\mathrm{Hom}^S_{\mathcal{A}}(H, \_)$, it is clear that $\mathrm{Hom}^S_{\mathcal{A}}(H, \_)|_{\mathcal{B}}$ is also faithful and exact.

We have to show that $\mathrm{Hom}^S_{\mathcal{A}}(H, \_)|_{\mathcal{B}}$ is full, i.e. for any two objects B and C in $\mathcal{B}$ and for any S-linear morphism $f : \mathrm{Hom}^S_{\mathcal{A}}(H, B) \to \mathrm{Hom}^S_{\mathcal{A}}(H, C)$ (a morphism in $\mathrm{MOD}_{S\mathcal{U}}$), there must exist a morphism $g : B \to C$ in $\mathcal{B}$ such that $g_* = f$.

Due to α_B and α_C being epimorphisms, it is easily shown that $\beta_B : H \to B$ and $\beta_C : H \to C$ are epimorphisms. Therefore the upper row of the following diagram is exact since $\mathrm{Hom}^S_{\mathcal{A}}(H, \_)$ is exact due to (a):

$$\begin{array}{ccccc} 0 & \longleftarrow & \mathrm{Hom}^S_{\mathcal{A}}(H, C) & \xleftarrow{\beta_{C*}} & S \\ & & {\scriptstyle f \circ \beta_{B*}} \big\uparrow & \nearrow & \\ & & S & \xrightarrow{\quad v \quad} & \end{array} .$$

Then due to (**3.1.4**) we have that S is projective, implying the existence of an S-linear mapping $v : S \to S$ such that the above diagram commutes. Since v is S-linear, it is fully characterized by the mapping $1_H \mapsto u \in S$, implying that $u_* = v$.

We claim that there exists a morphism $g : B \to C$ in $\mathcal{B}$ such that the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\beta_B) & \xrightarrow{\iota_B} & H & \xrightarrow{\beta_B} & B & \longrightarrow & 0 \\ & & & & {\scriptstyle u} \big\downarrow & & \big\downarrow {\scriptstyle g} & & \\ & & & & H & \xrightarrow{\beta_C} & C & \longrightarrow & 0 \end{array} ,$$

commutes, whereby the rows are exact. It is enough to show that $\beta_C \circ u \circ \iota_B = 0_{\mathrm{Ker}(\beta_B),0}$ since β_B is the cokernel of ι_B, therefore it is enough to show that $(\beta_C \circ u \circ \iota_B)_* = \beta_{C*} \circ u_* \circ \iota_{B*}$ is the zero morphism since $\mathrm{Hom}^S_{\mathcal{A}}(H, \_)$ is faithful. We have $\beta_{C*} \circ u_* = \beta_{C*} \circ v = f \circ \beta_{B*}$ and thus

$$\beta_{C*} \circ u_* \circ \iota_{B*} = f \circ \beta_{B*} \circ \iota_{B*} = 0_{\mathrm{Ker}(\beta_{B*}), \mathrm{Hom}^S_{\mathcal{A}}(H,B)},$$

since $\beta_B \circ \iota_B$ is the zero morphism. g is a morphism in $\mathcal{A}$ and automatically also a morphism in $\mathcal{B}$ since $\mathcal{B}$ is a full $\mathcal{U}$-subcategory of $\mathcal{A}$.

In particular since $g_* \circ \beta_{B*} = \beta_{C*} \circ u_* = \beta_{C*} \circ v = f \circ \beta_{B*}$, we have that $g_* = f$ since β_{B*} is an epimorphism (as β_B is an epimorphism in $\mathcal{A}$). $\square$

3.4 Grothendieck Abelian Categories

Let $\mathcal{U}$ be a Grothendieck universe and $\mathcal{A}$ be an abelian $\mathcal{U}$-category.

3.4.1 Definition (Grothendieck Abelian Categories) (II.3.2.1 [Mor20]**):** $\mathcal{A}$ is a *Grothendieck abelian $\mathcal{U}$-category* if:

(i) $\mathcal{A}$ has a generator G.

(ii) All $\mathcal{U}$-small colimits exist in $\mathcal{A}$.

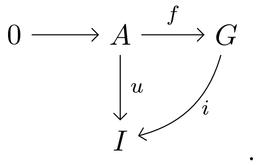
(iii) For a $\mathcal{U}$-small filtered category $\mathcal{I}$, the colimit functor $\mathrm{colim}_{\mathcal{I}} : \mathrm{FUNC}(\mathcal{I}, \mathcal{A}) \to \mathcal{A}$ is exact.

Importantly, it can be shown with difficulty that a Grothendieck abelian $\mathcal{U}$-category has all $\mathcal{U}$-small limits, see ([NCa21b]).

3.4.2 Examples (Grothendieck Abelian Categories) (II.3.2.2(1) [Mor20]**):** We have already proven that for a $\mathcal{U}$-ring R, ${}_R\mathrm{MOD}_{\mathcal{U}}$ and $\mathrm{MOD}_{R\mathcal{U}}$ are Grothendieck abelian $\mathcal{U}$-categories. They have a generator R due to (**3.2.8**(a)), they contain all $\mathcal{U}$-small colimits due to (**1.5.12**(b)), and $\mathrm{colim}_{\mathcal{I}} : \mathrm{FUNC}(\mathcal{I}, \mathcal{A}) \to \mathcal{A}$ is exact for all $\mathcal{U}$-small and filtered categories $\mathcal{I}$ due to (**2.4.12**).

The following lemma gives a criterion for injective objects in Grothendieck abelian $\mathcal{U}$-categories, where less conditions have to be checked as compared to (**3.1.2**(a)). It generalizes the *Baer criterion* used in categories of modules ([NCa21a]).

3.4.3 Lemma (Criterion for Injectives) (079G [JC21]**), (II.3.2.3** [Mor20]**):** Let $\mathcal{A}$ be a Grothendieck abelian $\mathcal{U}$-category with a generator G. Then an object I in $\mathcal{A}$ is injective if and only if for all monomorphisms $f : A \to G$ and morphisms $u : A \to I$ in $\mathcal{A}$, there exists an $i : G \to I$ (not necessarily unique) so that $i \circ f = u$, i.e. the following diagram commutes:



Proof: For forward implication: This implication has already been shown in (**3.1.2**(a)) with particular monomorphisms $f : A \to G$ in $\mathcal{A}$ instead of monomorphisms $f : A \to B$ in $\mathcal{A}$.

For backward implication: It is enough to show that the condition in (**3.1.2**(a)) is fulfilled for all monomorphisms $f : A \to B$ and morphisms $u : A \to I$ in $\mathcal{A}$. Let C be the set:

$$C = \{(B_1, \iota_1, f_1, u_1 : B_1 \to I) | (B_1, \iota_1) \in \mathrm{Sub}(B), (A, f_1) \in \mathrm{Sub}(B_1), u = u_1 \circ f_1\} / \sim,$$

defined with the following equivalence relation: $(B_1, \iota_1, f_1, u_1) \sim (B_2, \iota_2, f_2, u_2)$ if and only if (B_1, ι_1) and (B_2, ι_2) are the same subobjects via an isomorphism $\varphi : B_1 \to B_2$, and (A, f_1) and (A, f_2) are the same subobjects, and $u_1 = u_2 \circ \varphi$, $f_2 = \varphi \circ f_1$. It is easy to check that this defines an equivalence relation.

(⋆) Then for any two elements (B_1, ι_1, f_1, u_1) and (B_2, ι_2, f_2, u_2) in C, we define the order relation: $(B_1, \iota_1, f_1, u_1) \leq (B_2, \iota_2, f_2, u_2)$ if and only if $B_1 \subset B_2$, i.e. there exists a monomorphism $\iota_{1,2} : B_1 \to B_2$ with $\iota_1 = \iota_2 \circ \iota_{1,2}$, and $f_2 = \iota_{1,2} \circ f_1$ and furthermore $u_1 = u_2 \circ \iota_{1,2}$. It is easy to check that this defines a well-defined order relation on the equivalence classes.

We want to show that C has a maximal element and since we are operating in an extension of *Zermelo-Fraenkel set theory with the axiom of choice*, we can use *Zorn's lemma* as seen in ([Bel15]). Thus, it would be enough to show that every totally-ordered subset, i.e. chain $K \subset C$, has an upper bound in C. Let $K = \{(B_m, \iota_m, f_m, u_m) \in C\}_{m \in M}$ be a nonempty chain of C (the case $K = \emptyset$ is trivial), which makes $\{B_m\}_{m \in M}$ a totally-ordered collection of subobjects of A due to (**2.5.3**). Due to (**3.2.6**), we have that $\{B_m\}_{m \in M}$ is $\mathcal{U}$-small and thus $M \in \mathcal{U}$.

We define the inclusion functor $D : \mathcal{M} \to \mathcal{A}$ whereby $\mathcal{M}$ is the $\mathcal{U}$-small category with elements $\{B_m\}_{m \in M}$ as objects and the subobject monomorphisms $\{\iota_{m,n}\}_{m,n \in M, B_m \subset B_n}$ being the morphisms of $\mathcal{M}$. Since $\mathcal{A}$ is a Grothendieck abelian $\mathcal{U}$-category, the colimit $(B', (\iota'_m : B_m \to B')_{m \in M})$ exists in $\mathcal{A}$. Furthermore, it is clear that $(\iota_m : B_m \to B)_{m \in M}$ and $(u_m : B_m \to I)_{m \in M}$ define D-cocones and thus there exists canonical morphisms $\iota' : B' \to B$, $u' : B' \to I$ induced by the colimits. We define $f' : A \to B'$ as $f' = \iota'_m \circ f_m$ for any $m \in M$, this is independent of the choice of $m \in M$, since for $B_m \subset B_n$ we have $f' = \iota'_n \circ f_n = \iota'_n \circ \iota_{m,n} \circ f_m = \iota'_m \circ f_m$.

Since $(\iota_m)_{m \in M}$ consists of monomorphisms in $\mathcal{A}$ and since we have $\iota_m = \iota' \circ \iota'_m$ for all $m \in M$, $(\iota'_m)_{m \in M}$ must be monomorphisms and thus $(B_m)_{m \in M}$ are subobjects of B', $A \in \mathrm{Sub}(B')$ also as $A \subset B_m \subset B'$ for any $m \in M$. This also implies that $f' = \iota'_m \circ f_m$ is a monomorphism for all $m \in M$ since ι'_m and f_m are monomorphisms.

As $\mathcal{M}$ is the category induced by a totally-ordered set $\{B_m\}_{m \in M}$, it is clearly $\mathcal{U}$-small filtered and thus $\mathrm{colim}_{\mathcal{M}} : \mathrm{FUNC}(\mathcal{M}, \mathcal{A}) \to \mathcal{A}$ is an exact functor that sends monomorphisms to monomorphisms due to (**2.4.8**). Since the morphisms $(\iota_m : B_m \to B)_{m \in M}$ define a natural monomorphism v between D and $\triangle(B)$ within $\mathrm{Hom}_{\mathrm{FUNC}(\mathcal{M},\mathcal{A})}(D, \triangle(B))$, we have that $\mathrm{colim}_{\mathcal{M}} v = \iota'$ is a monomorphism and B' is a subobject of B. Since $u' \circ f' = u' \circ \iota'_m \circ f_m = u_m \circ f_m = u$ for any $m \in M$, we therefore have $(B', \iota', f', u') \in C$, which is an upper bound of K.

Let (B'', ι'', f'', u'') be a maximal element of C, we then claim that the monomorphism $\iota'' : B'' \to B$ is an isomorphism. Assume that ι'' is not an isomorphism, then since G is a generator, it follows due to (**3.2.5**) that the left composition is $\iota''_* : \mathrm{Hom}_{\mathcal{A}}(G, B'') \to \mathrm{Hom}_{\mathcal{A}}(G, B)$ is not a bijection (as it cannot be a surjection and it is already an injection). There must exist a nonzero morphism $\varphi : G \to B$ that does not factor through ι'', implying that $\mathrm{Im}(\varphi)$ is not a subobject of B''. Let $X = B'' \cap \mathrm{Im}(\varphi)$ as a subobject of B and B'' with the monomorphisms $\psi : X \to B$ and $\psi'' : X \to B''$. Let $Y = X \times_B G$ be the fiber product with projections p_X and p_G induced from φ and ψ. We have the cartesian diagram:

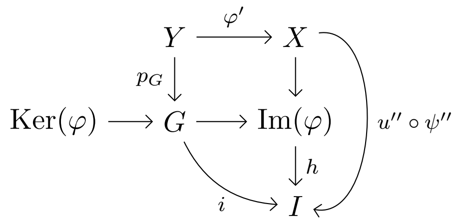
$$\begin{array}{ccc} Y & \xrightarrow{p_G} & G \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow{\psi} & B \end{array},$$

whereby ψ being a monomorphism implies that p_G is a monomorphism due to (**2.6.3**(a)), thus Y is a subobject of G. We then set $\varphi' = p_X : Y \to X$ and let $g = u'' \circ \psi'' \circ \varphi' : Y \to I$. Then due to our original condition, there exists an $i : G \to I$ such that the diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & Y & \xrightarrow{p_G} & G \\ & & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle i} & \\ & & I & & \end{array},$$

commutes. Since $\mathrm{Ker}(\varphi)$ is a subobject of Y (due to p_G being a monomorphism and pullback properties) and since $\mathrm{Ker}(\varphi)$ is a subobject of $\mathrm{Ker}(g)$ (due to pullback properties), we have

$\mathrm{Ker}(\varphi) \subset \mathrm{Ker}(g) \subset \mathrm{Ker}(i)$. Therefore $\mathrm{Ker}(\varphi) \to G \xrightarrow{i} I$ is the zero morphism, which induces a morphism $h : \mathrm{Im}(\varphi) \to I$ as $\mathrm{Im}(\varphi)$ is the cokernel of $\mathrm{Ker}(\varphi) \to G$. We then have the commutativity of the diagram:



.

Clearly restricting h and u'' onto $X = B'' \cap \mathrm{Im}(\varphi)$ gives us the same morphism and thus due to (**2.6.4**), there must exist a morphism $\gamma : B'' \cup \mathrm{Im}(\varphi) \to I$ that extends both h and u'', and since $\mathrm{Im}(\varphi)$ is not a subobject of B'', B'' is a nontrivial subobject of $B'' \cup \mathrm{Im}(\varphi)$, which is still a subobject of B. γ and $B'' \cup \mathrm{Im}(\varphi)$ together clearly define an element of C which is larger than (B'', ι'', f'', u'') in our order relation, which is a contradiction to its maximality.

Since $\iota'' : B'' \to B$ is an isomorphism, we choose $\iota = u'' \circ \iota''^{-1} : B \to I$. Due to (B'', ι'', f'', u'') being maximal in C, we have $(A, f, 1_A, u) \leq (B'', \iota'', f'', u'')$ which implies $f = \iota'' \circ f''$ due to the definition of the order relation in $(\star)$ (in that context we can directly read $\iota_{1,2} = f''$ from the equalities). It then follows that:

$$\iota \circ f = u'' \circ \iota''^{-1} \circ f = u'' \circ \iota''^{-1} \circ \iota'' \circ f'' = u'' \circ f'' = u,$$

and therefore I fulfills the condition from (**3.1.2**(a)). ☐

3.4.4 Definitions (Well-Ordered Sets and Cofinalities) (II.3.2.5 [Mor20]**):**

(a) A set I is *well-ordered* if it is a totally-ordered set such that every subset $J \subset I$ has a well-defined smallest element.

(b) Let I be a well-ordered set. A subset $J \subset I$ is *cofinal* if for every $i \in I$ there exists an element $j \in J$ so that $i \leq j$. Furthermore, the *cofinality* of I is the smallest cardinality of all the cofinal subsets of I.

3.4.5 Lemma (Cofinalities) (II.3.2.6 [Mor20]**):** Let I be a $\mathcal{U}$-small set, then there exists a well-ordered $\mathcal{U}$-small set γ, whose elements we call ordinals and whose cofinality is strictly greater than the cardinality of I.

Proof: See reference. ☐

3.4.6 Lemma (Enough Injectives) (079H [JC21]**), (II.3.2.4** [Mor20]**):** A Grothendieck abelian $\mathcal{U}$-category $\mathcal{A}$ has enough injectives. More precisely, there exists a functor $I : \mathcal{A} \to \mathcal{A}$ and a natural transformation $\iota : 1_{\mathcal{A}} \to I$ in $\mathrm{F{\scriptsize UNC}}(\mathcal{A}, \mathcal{A})$ such that for all objects A in $\mathcal{A}$, IA is an injective object and $\iota A : A \to IA$ is a monomorphism.

Proof: Let G be a generator of $\mathcal{A}$ and A be an object of $\mathcal{A}$. The objects $\coprod_{B \in \mathrm{Sub}(G)} \coprod_{f \in \mathrm{Hom}_{\mathcal{A}}(B,A)} B$ and $\coprod_{B \in \mathrm{Sub}(G)} \coprod_{f \in \mathrm{Hom}_{\mathcal{A}}(B,A)} G$ exist in $\mathcal{A}$ as they are $\mathcal{U}$-small indexed coproducts. This is because the set of subobjects $\mathrm{Sub}(G)$ is $\mathcal{U}$-small due to (**3.2.6**), and furthermore for all $B \in \mathrm{Sub}(G)$, we have that $\mathrm{Hom}_{\mathcal{A}}(B, A)$ is a $\mathcal{U}$-set due to (**1.2.5**(a)).

We define a morphism $\iota : \coprod_{B \in \mathrm{Sub}(G)} \coprod_{f \in \mathrm{Hom}_{\mathcal{A}}(B,A)} B \to \coprod_{B \in \mathrm{Sub}(G)} \coprod_{f \in \mathrm{Hom}_{\mathcal{A}}(B,A)} G$ as follows: For any subobject $(B, \iota_B) \in \mathrm{Sub}(G)$ with the corresponding monomorphism $\iota_B : B \to G$, we define a morphism $\iota'_B : B \to \coprod_{B \in \mathrm{Sub}(G)} \coprod_{f \in \mathrm{Hom}(B,A)} G$ induced by the morphisms $(\iota_B : B \to G)_{f \in \mathrm{Hom}_{\mathcal{A}}(B,A)} \cup (0_{B,G})_{B' \in \mathrm{Sub}(G), B' \neq B, f \in \mathrm{Hom}_{\mathcal{A}}(B',A)}$. We then induce the morphism ι from the morphisms $(\iota'_B : B \to \coprod_{B \in \mathrm{Sub}(G)} \coprod_{f \in \mathrm{Hom}(B,A)} G)_{B \in \mathrm{Sub}(G), f \in \mathrm{Hom}_{\mathcal{A}}(B,A)}$. We also define $a : \coprod_{B \in \mathrm{Sub}(G)} \coprod_{f \in \mathrm{Hom}_{\mathcal{A}}(B,A)} B \to A$ as the morphism induced by the morphisms

$(f : B \to A)_{B\in \mathrm{Sub}(G), f\in \mathrm{Hom}_{\mathcal{A}}(B,A)}$. Then we define the object φA and the morphisms φ_A and b as the pushout in the following diagram in $\mathcal{A}$:

$$\begin{array}{ccc} \coprod_{B\in \mathrm{Sub}(G)} \coprod_{f\in \mathrm{Hom}_{\mathcal{A}}(B,A)} B & \xrightarrow{a} & A \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \varphi_A} \\ \coprod_{B\in \mathrm{Sub}(G)} \coprod_{f\in \mathrm{Hom}_{\mathcal{A}}(B,A)} G & \xrightarrow{b} & \varphi A \end{array} .$$

The existence of this pushout is guaranteed due to (**2.3.5**) since $\mathcal{A}$ is an abelian $\mathcal{U}$-category.

We then define a functor $\varphi : \mathcal{A} \to \mathcal{A}$ through an object mapping $A \mapsto \varphi A$. Since our construction of φ is induced by a colimit functor, the mapping $A \mapsto \varphi A$ canonically extends to a functorial one through colimit morphisms. As ι is induced from monomorphisms, it is itself a monomorphism and as we have a cocartesian square above, we have that $\varphi_A : A \to \varphi A$ is also a monomorphism due to (**2.6.3**(d)). Thus, the natural transformation $\psi : 1_A \to \varphi$ defined by the morphisms $(\varphi_A : A \to \varphi A)_{A\in \mathrm{Ob}(\mathcal{A})}$ is a monomorphism due to (**1.3.4**(a)).

Since $\mathrm{Sub}(G)$ is $\mathcal{U}$-small due to (**3.2.6**), there exists a well-ordered $\mathcal{U}$-small set γ such that γ has a cofinality strictly greater than the cardinality of $\mathrm{Sub}(G)$ due to (**3.4.5**). As γ is well-ordered, γ contains a minimum element which we label as 0. Furthermore, there exists an ordinal $\alpha \in \gamma$ such that the subset $\gamma_{\leq\alpha} \subset \gamma$ of ordinals lesser or equal to α has a cardinality strictly greater than the cofinality of $\mathrm{Sub}(G)$.

We will use transfinite induction on γ, beginning at 0 and ending at α, to recursively define functors $\varphi_\beta : \mathcal{A} \to \mathcal{A}$ for $\beta \in \gamma_{\leq\alpha}$, with $\varphi_0 = 1_{\mathcal{A}}$. Furthermore, we define natural transformations $\psi_{\beta_1,\beta_2} : \varphi_{\beta_1} \to \varphi_{\beta_2}$ whereby $\beta_1, \beta_2 \in \gamma_{\leq\alpha}$, $\beta_1 \leq \beta_2$, with $\psi_{\beta,\beta}$ being the identity on φ_β for $\beta \in \gamma_{\leq\alpha}$. For every time we increase the ordinal of $\beta \in \gamma_{\leq\alpha}$, we also claim that for all ordinals $\beta_1 \leq \beta_2 \leq \beta_3 \leq \beta$ in $\gamma_{\leq\alpha}$, $\psi_{\beta_1,\beta_2} : \varphi_{\beta_1} \to \varphi_{\beta_2}$ is a monomorphism and we have $\psi_{\beta_2,\beta_3} \circ \psi_{\beta_1,\beta_2} = \psi_{\beta_1,\beta_3}$.

For transfinite induction, base case: We have $\varphi_0 = 1_{\mathcal{A}}$ and $\psi_{0,0} : 1_{\mathcal{A}} \to 1_{\mathcal{A}}$ as the identity. We then define $\varphi_1 = \varphi$, $\psi_{0,1} = \psi : 1_{\mathcal{A}} \to \varphi_1$ with $\psi_{1,1} : \varphi_1 \to \varphi_1$ being the identity. We know that $\psi_{0,0}$ and $\psi_{0,1}$ and $\psi_{1,1}$ are monomorphisms. It is also easy to check that $\psi_{\beta_2,\beta_3} \circ \psi_{\beta_1,\beta_2} = \psi_{\beta_1,\beta_3}$ for all ordinals $0 \leq \beta_1 \leq \beta_2 \leq \beta_3 \leq 1$.

For transfinite induction, successor ordinals: If $\beta + 1 \in \gamma_{\leq\alpha}$ is a successor ordinal to $\beta \in \gamma_{\leq\alpha}$, i.e. $\beta+1$ is the minimum of the subset $\gamma_{>\beta} \subset \gamma$ of ordinals greater than β. We set $\varphi_{\beta+1} = \varphi \circ \varphi_\beta$ and for all $\beta_1 \leq \beta$ we set $\psi_{\beta_1,\beta+1} = \psi \circ \psi_{\beta_1,\beta}$, with $\psi_{\beta+1,\beta+1}$ being the identity on $\varphi_{\beta+1}$. $\psi_{\beta+1,\beta+1}$ is clearly a monomorphism, and since ψ and $\psi_{\beta_1,\beta}$ are monomorphisms due to the induction hypothesis, $\psi_{\beta_1,\beta+1}$ must also be a monomorphism. With the induction hypothesis, it is clear that ψ_{β_1,β_2} is a monomorphism for all $\beta_1 \leq \beta_2 \leq \beta + 1$. The equality $\psi_{\beta_2,\beta_3} \circ \psi_{\beta_1,\beta_2} = \psi_{\beta_1,\beta_3}$ is true for the ordinals $\beta_1 \leq \beta_2 \leq \beta_3 \leq \beta$ due to the induction hypothesis, the equality also trivially clear if $\beta_1 = \beta + 1$, the remaining cases $\beta_3 = \beta + 1$, $\beta_2 = \beta + 1$ follow from the definitions and the associativity of composition.

For transfinite induction, limit ordinals: Standard induction is not enough, we may not yet be able to reach the ordinal α and construct φ_α and ψ_α. To resolve this, let $\beta \in \gamma_{\leq\alpha}$ be a limit ordinal, which are ordinals that we cannot obtain from smaller ordinals $\beta_1 < \beta$ by recursively applying successor ordinals. We then define $\mathcal{I}_{<\beta}$ as the $\mathcal{U}$-small category whose objects are the subset $\gamma_{<\beta} \subset \gamma$ of ordinals lesser than β, and for two objects β_1, β_2 in $\mathcal{I}_{<\beta}$, we have a morphism $\iota : \beta_1 \to \beta_2$ if and only if $\beta_1 \leq \beta_2$. $\mathcal{I}_{<\beta}$ is thus clearly a $\mathcal{U}$-small category as γ is $\mathcal{U}$-small. Due to the induction hypothesis and our construction of ψ_{β_1,β_2}, we have that $D_{<\beta} : \mathcal{I}_{<\beta} \to \mathrm{FUNC}(\mathcal{A}, \mathcal{A})$, $\beta_1 \mapsto \varphi_{\beta_1}$, $(\iota : \beta_1 \to \beta_2) \mapsto \psi_{\beta_1,\beta_2}$ forms a functor.

As $\mathcal{A}$ contains all $\mathcal{U}$-small colimits, we have that $\mathrm{FUNC}(\mathcal{A}, \mathcal{A})$ contains all $\mathcal{U}$-small colimits due to (**I.5.3.1** [Mor20]), we then define $\varphi_\beta = \mathrm{colim}_{\mathcal{I}_{<\beta}} D_{<\beta} : \mathcal{A} \to \mathcal{A}$ and define $\psi_{\beta_1,\beta} : \varphi_{\beta_1} \to \varphi_\beta$ as the corresponding morphisms from the colimit cone. Thus, the equalities $\psi_{\beta_2,\beta_3} \circ \psi_{\beta_1,\beta_2} = \psi_{\beta_1,\beta_3}$ for all ordinals $\beta_1 \leq \beta_2 \leq \beta_3 \leq \beta$ follow from the induction hypothesis and from the properties of the colimit cone. Since filtered $\mathcal{U}$-small colimits are exact in $\mathcal{A}$, they must also be exact in $\mathrm{Func}(\mathcal{A}, \mathcal{A})$ due to how kernels and cokernels in $\mathrm{Func}(\mathcal{A}, \mathcal{A})$ can be found object-wise. Since $\mathcal{I}_{<\beta}$ is a $\mathcal{U}$-small filtered category, we then have that $\mathrm{colim}_{\mathcal{I}_{<\beta}}$ maps monomorphisms to monomorphisms. For the natural transformation $E_{\beta_1} : \triangle(\varphi_{\beta_1}) \to D_{<\beta}$ given via the monomorphisms $(\psi_{\beta_1,\beta_2} : \varphi_{\beta_1} \to \varphi_{\beta_2})_{\beta_2 \in \mathrm{Ob}(\mathcal{I}_{<\beta})}$, we have that E_{β_1} is a monomorphism and thus $\mathrm{colim}_{\mathcal{I}_{<\beta}} E_{\beta_1} : \varphi_{\beta_1} \to \varphi_\beta$ is a monomorphism. $\mathrm{colim}_{\mathcal{I}_{<\beta}} E_{\beta_1}$ coincides with $\psi_{\beta_1,\beta}$ due to the construction of the colimit functor and thus $\psi_{\beta_1,\beta}$ is a monomorphism. With $\psi_{\beta,\beta}$ being the identity, we then have that all morphisms ψ_{β_1,β_2} for $\beta_1, \beta_2 \leq \beta$ are monomorphisms.

For definition of I and properties: We then set $I = \varphi_\alpha : \mathcal{A} \to \mathcal{A}$ as our desired functor, for which we have already shown that the mappings $(\iota A = \psi_{0,\alpha} A : A \mapsto IA)_{A \in \mathrm{Ob}(\mathcal{A})}$ are monomorphisms. We still have to show that IA is injective for all objects A in $\mathcal{A}$. To do this, let A be a fixed object in $\mathcal{A}$ and let $f : C \to G$ be any monomorphism and $u : C \to IA$ be any morphism in $\mathcal{A}$, due to (**3.4.3**) it suffices to show that there exists $i : G \to IA$ such that $i \circ f = u$. Since IA can be described as the colimit of a diagram from $\mathcal{I}_{\leq\alpha}$ to $\mathcal{A}$ and since $\mathrm{Sub}(C)$ injects into $\mathrm{Sub}(G)$, we have with (**II.3.2.7** [Mor20]) that $u : C \to IA$ factors into two morphisms $v : C \to \varphi_\beta A$, $\psi_{\beta,\alpha} A : \varphi_\beta A \to IA$ with $u = \psi_{\beta,\alpha} A \circ v$ for an ordinal $\beta < \alpha$. Due to the definition of $\varphi \circ \varphi_\beta A = \varphi_{\beta+1} A$ as a pushout, we have that v is part of the morphisms that induce a in the following diagram:

$$\begin{array}{ccc} \coprod_{B \in \mathrm{Sub}(G)} \coprod_{f \in \mathrm{Hom}_{\mathcal{A}}(B, \varphi_\beta A)} B & \xrightarrow{a} & \varphi_\beta A \\ {\scriptstyle \iota}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_{\varphi_\beta A}} \\ \coprod_{B \in \mathrm{Sub}(G)} \coprod_{f \in \mathrm{Hom}_{\mathcal{A}}(B, \varphi_\beta A)} G & \xrightarrow{b} & \varphi_{\beta+1} A \end{array},$$

and thus since the diagram commutes, there exists a $v' : G \to \varphi_{\beta+1} A$ that induces b such that v' extends $\varphi_{\varphi_\beta A} \circ v : C \to \varphi_{\beta+1} A$, with respect to the monomorphism $f : C \to G$. Since v' extends $\varphi_{\varphi_\beta A} \circ v$, we have that $i = \psi_{\beta+1,\alpha} A \circ v' : G \to IA$ extends $u = \psi_{\beta,\alpha} A \circ v = \psi_{\beta+1,\alpha} A \circ \varphi_{\varphi_\beta A} \circ v$ with respect to $f : C \to G$. Thus, we have $i \circ f = u$ and IA is injective for all objects A in $\mathcal{A}$ as claimed and $\mathcal{A}$ has enough injectives. ☐

3.4.7 Lemma (Injective Cogenerators) (II.3.2.8 [Mor20]**):** A Grothendieck abelian $\mathcal{U}$-category $\mathcal{A}$ has an injective cogenerator.

Proof: Since $\mathcal{A}$ has $\mathcal{U}$-small limits and due to (**3.2.9**), it suffices to show that there exists an injective object I in $\mathcal{A}$ such that for every nonzero object A in $\mathcal{A}$, there exists a nonzero morphism $\alpha_A : A \to I$. With G as a generator of $\mathcal{A}$, we have that $\mathrm{Sub}(G)$ is $\mathcal{U}$-small due to (**3.2.6**), therefore we have an object $C = \coprod_{B \in \mathrm{Sub}(G)} G/B$ in $\mathcal{A}$, where G/B are the cokernels of monomorphisms $f : B \to G$. As constructed in (**3.4.6**), there exists a canonical monomorphism $\iota : C \to IC$ with IC being an injective object.

We claim that IC is also a cogenerator. As G is a generator, we have that $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ is faithful with the help of (**3.2.4**). For any nonzero object A in $\mathcal{A}$, we have that $1_A \neq 0_{A,A}$ due to (**2.1.3**), implying that $(1_A)_* : \mathrm{Hom}_{\mathcal{A}}(G, A) \to \mathrm{Hom}_{\mathcal{A}}(G, A)$ is nonzero due to $\mathrm{Hom}_{\mathcal{A}}(G, \_)$ being faithful. Therefore $\mathrm{Hom}_{\mathcal{A}}(G, A)$ is not a trivial group and there exists a nonzero morphism $f_A : G \to A$. This implies that $\mathrm{Coim}(f_A) \cong G/\mathrm{Ker}(f_A)$ is a nonzero object in $\mathcal{A}$, which implies that C is a nonzero object.

$f_A : G \to A$ induces a nonzero monomorphism $g_A : \mathrm{Coim}(f_A) \cong G/\mathrm{Ker}(f_A) \to A$ due to the decomposition as seen in (**2.3.1**). Due to the construction of C as a coproduct, there exists a nonzero morphism $\iota_{G/\mathrm{Ker}(f_A)} : G/\mathrm{Ker}(f_A) \to C$. Since $\mathcal{A}$ has enough injectives due to (**3.4.6**), there exists a nonzero monomorphism $\iota C : C \to IC$, which makes $\iota C \circ \iota_{G/\mathrm{Ker}(f_A)} : G/\mathrm{Ker}(f_A) \to IC$ a nonzero morphism. g_A and $\iota C \circ \iota_{G/\mathrm{Ker}(f_A)}$ give us a morphism $\alpha_A : A \to IC$ such that $\alpha_A \circ g_A = \iota C \circ \iota_{G/\mathrm{Ker}(f_A)}$ due to (**3.1.2**(a)) as IC is injective. Since g_A and $\iota C \circ \iota_{G/\mathrm{Ker}(f_A)}$ are nonzero, α_A is also nonzero, otherwise $\alpha_A = 0_{A,IC}$ would imply that $\alpha_A \circ g_A = \iota C \circ \iota_{G/\mathrm{Ker}(f_A)}$ is a zero morphism, which is a contradiction. Thus the claim follows from (**3.2.9**). ☐

3.4.8 Lemma (Modules have Enough Injectives and Projectives) (II.3.2.9 [Mor20]**):** Let R be a $\mathcal{U}$-ring. The abelian $\mathcal{U}$-categories ${}_R\text{MOD}_{\mathcal{U}}$ and $\text{MOD}_{R\mathcal{U}}$ have enough injective and projective objects.

Proof: Due to (**1.3.9**), it is enough to prove the lemma for ${}_R\text{MOD}_{\mathcal{U}}$.

For enough injectives: Due to (**3.4.2**), we have that ${}_R\text{MOD}_{\mathcal{U}}$ is a Grothendieck abelian $\mathcal{U}$-category and thus due to (**3.4.6**), we have that ${}_R\text{MOD}_{\mathcal{U}}$ has enough injectives.

For enough projectives: We claim that all free modules, i.e. modules with a basis, in ${}_R\text{MOD}_{\mathcal{U}}$ are projective. Let P be any free module, then P is isomorphic to $P \cong \bigoplus_{i\in I} R$ for an index set $I \in \mathcal{U}$. Due to (**3.1.3**(b)), it suffices to show that any exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} \bigoplus_{i\in I} R \to 0$ in ${}_R\text{MOD}_{\mathcal{U}}$ splits. We must find a morphism $h : \bigoplus_{i\in I} R \to B$ such that $g \circ h = 1_{\bigoplus_{i\in I} R}$. Since g is an R-linear epimorphism due to (**2.4.2**(b)), there exists a collection of elements $(b_i)_{i\in I}$ in B such that $g(b_i) = 1_i$ with 1_i as the multiplicative unit 1_i on the i-coordinate of $\bigoplus_{i\in I} R$. We then define $h : \bigoplus_{i\in I} R \to B$ as the R-linear mapping generated from the mappings $h(1_i) = b_i$ for $i \in I$. Then it is clear that $g \circ h = 1_{\bigoplus_{i\in I} R}$ and therefore P is projective.

For any module M in ${}_R\text{MOD}_{\mathcal{U}}$, we define $P_M = \bigoplus_{m\in M} R$ as the free module on M which is projective. We then have the epimorphism $\varphi_M : P_M \to M$ generated from the mappings $\varphi_M(1_m) = m$ for $m \in M$. Therefore, ${}_R\text{MOD}_{\mathcal{U}}$ has enough projectives. □

4 Sheaves and their Generalizations

We can now study sheaves and generalize them in more detail. Although it is valuable to study sheaves on topological spaces and how sheafification and separation is constructed for such sheaves, as seen in (**III.1** [Mor20]), *Mitchell's embedding theorem* requires the generalization of sheaves on Grothendieck pretopologies and how sheafification works on such sheaves, as seen in (**III.2** [Mor20]). Let $\mathcal{U}$ be a Grothendieck universe.

4.1 Introduction to Sheaves

4.1.1 Definitions (Concrete and Good Concrete Categories) (1.19 [Bö21]**), (0073** [JC21]**):** Let $\mathcal{D}$ be a $\mathcal{U}$-category with a so-called forgetful functor $\text{For} : \mathcal{D} \to \text{SET}_{\mathcal{U}}$. $\mathcal{D}$, together with For, is called a *concrete $\mathcal{U}$-category*, if:

(i) For is faithful.

$\mathcal{D}$, together with For, is furthermore called a *good concrete $\mathcal{U}$-category*, if it is a concrete $\mathcal{U}$-category and:

(ii) For is conservative.

(iii) $\mathcal{D}$ has all $\mathcal{U}$-small limits and For commutes with these limits.

(iv) $\mathcal{D}$ has all $\mathcal{U}$-small filtered colimits and For commutes with these filtered colimits.

4.1.2 Examples (Good Concrete Categories):

(a) Many $\mathcal{U}$-categories we have encountered whose objects are "sets with extra structure" are concrete $\mathcal{U}$-categories, e.g. $\text{RNG}_{\mathcal{U}}$, ${}_R\text{MOD}_{\mathcal{U}}$ for a $\mathcal{U}$-ring R.

(b) $\text{SET}_{\mathcal{U}}$ with the identity functor $\text{For} : \text{SET}_{\mathcal{U}} \to \text{SET}_{\mathcal{U}}$ is clearly a good concrete $\mathcal{U}$-category.

(c) For a $\mathcal{U}$-ring R, the categories ${}_R\text{MOD}_{\mathcal{U}}$ and $\text{MOD}_{R\mathcal{U}}$ with their respective forgetful functors are good concrete categories. (**4.1.1**(i)) follows since morphisms in ${}_R\text{MOD}_{\mathcal{U}}$ and $\text{MOD}_{R\mathcal{U}}$ are uniquely identified by their underlying functions. (**4.1.1**(ii)) follows since every module homomorphism that is a bijection must be a module isomorphism. (**4.1.1**(iii)) and (**4.1.1**(iv)) follow from (**1.5.12**) and (**1.5.17**).

(d) Analogously to ${}_R\textsc{Mod}_{\mathcal{U}}$ and $\textsc{Mod}_{R\mathcal{U}}$, the $\mathcal{U}$-categories $\textsc{Ab}_{\mathcal{U}}$ and $\textsc{Vec}_{K\mathcal{U}}$ for a $\mathcal{U}$-field K and $\textsc{Rng}_{\mathcal{U}}$ are also good concrete $\mathcal{U}$-categories.

4.1.3 Lemma (Identifying Morphisms with Underlying Functions) (1.19 [Bö21]**):** Let $\mathcal{D}$, together with For, be a good concrete $\mathcal{U}$-category. For a morphism $f : A \to B$ in $\mathcal{D}$, we have that f is a monomorphism (respectively epimorphism, isomorphism) if and only if $\mathrm{For}(f) : \mathrm{For}A \to \mathrm{For}B$ is a monomorphism (respectively epimorphism, isomorphism).

Proof: For backward implication: F clearly reflects isomorphisms as it is conservative. F is also faithful and thus reflects monomorphisms and epimorphisms due to (**3.2.3**).

For forward implication: If $f : A \to B$ is an isomorphism in $\mathcal{D}$, it must have an inverse morphism $g : B \to A$ that fulfills $g \circ f = 1_A$ and $f \circ g = 1_B$, this implies $\mathrm{For}(g) \circ \mathrm{For}(f) = 1_{\mathrm{For}A}$ and $\mathrm{For}(f) \circ \mathrm{For}(g) = 1_{\mathrm{For}B}$. Therefore, $\mathrm{For}(f)$ is also an isomorphism.

Let $f : A \to B$ be a monomorphism in $\mathcal{D}$, then we claim that the left composition $(\mathrm{For}(f))_* : \mathrm{Hom}_{\textsc{Set}_{\mathcal{U}}}(W, \mathrm{For}A) \to \mathrm{Hom}_{\textsc{Set}_{\mathcal{U}}}(W, \mathrm{For}B)$, $g \mapsto \mathrm{For}(f) \circ g$ is injective for all $\mathcal{U}$-sets W. Let $(A \times_B A, p_1, p_2)$ be the pullback of the morphism $f : A \to B$ applied twice, with the projection morphisms $p_1, p_2 : A \times_B A \to A$, this pullback is contained in $\mathcal{D}$ as a $\mathcal{U}$-small limit. As For commutes with $\mathcal{U}$-small limits, we have that $(\mathrm{For}(A \times_B A), \mathrm{For}(p_1), \mathrm{For}(p_2))$ is the pullback of $\mathrm{For}(f) : \mathrm{For}A \to \mathrm{For}B$ applied twice. Thus, for any two morphisms $g, h : W \to \mathrm{For}A$ in $\textsc{Set}_{\mathcal{U}}$ such that $\mathrm{For}(f) \circ g = \mathrm{For}(f) \circ h$, there must exist a morphism $\alpha : W \to \mathrm{For}(A \times_B A)$ such that $h = \mathrm{For}(p_1) \circ \alpha = \mathrm{For}(p_2) \circ \alpha = g$ due to the universal property of pullbacks. This makes $(\mathrm{For}(f))_*$ injective for all $\mathcal{U}$-sets W and thus $\mathrm{For}(f)$ is a monomorphism, i.e. injective.

Let $f : A \to B$ be an epimorphism in $\mathcal{D}$, then we claim that the right composition $(\mathrm{For}(f))^* : \mathrm{Hom}_{\textsc{Set}_{\mathcal{U}}}(\mathrm{For}B, W) \to \mathrm{Hom}_{\textsc{Set}_{\mathcal{U}}}(\mathrm{For}A, W)$, $h \mapsto h \circ \mathrm{For}(f)$ is injective for all $\mathcal{U}$-sets W. Let $(B \sqcup_A B, \iota_1, \iota_2)$ be the pushout of the morphism $f : A \to B$ applied twice, with the inclusion morphisms $\iota_1, \iota_2 : B \to B \sqcup_A B$, this pushout is contained in $\mathcal{D}$ as a filtered $\mathcal{U}$-small colimit. As For commutes with such colimits, we have that $(\mathrm{For}(B \sqcup_A B), \mathrm{For}(\iota_1), \mathrm{For}(\iota_2))$ is the pushout of $\mathrm{For}(f) : \mathrm{For}A \to \mathrm{For}B$ applied twice. Thus, for any two morphisms $g, h : \mathrm{For}B \to W$ in $\textsc{Set}_{\mathcal{U}}$ such that $g \circ \mathrm{For}(f) = h \circ \mathrm{For}(f)$, there must exist a morphism $\alpha : \mathrm{For}(B \sqcup_A B) \to W$ such that $h = \alpha \circ \mathrm{For}(\iota_1) = \alpha \circ \mathrm{For}(\iota_2) = g$ due to the universal property of pushouts. This makes $(\mathrm{For}(f))^*$ injective for all $\mathcal{U}$-sets W and thus $\mathrm{For}(f)$ is an epimorphism, i.e. surjective. □

4.1.4 Notes (Useful Notations): We will simplify some notation:

(a) Let (X, τ) be a $\mathcal{U}$-topological space and let $\mathcal{D}$ be a $\mathcal{U}$-category. Then we simplify our notation from (**1.4.1**) with $\textsc{PSh}(\textsc{Open}(X), \mathcal{D}) = \textsc{PSh}(X, \mathcal{D})$ and $\textsc{PSh}(\textsc{Open}(X), \textsc{Set}_{\mathcal{U}}) = \textsc{PSh}(\textsc{Open}(X)) = \textsc{PSh}(X)$. For a $\mathcal{U}$-ring R, we often write $\textsc{PSh}(X, {}_R\textsc{Mod}_{\mathcal{U}}) = \textsc{PSh}(X, R)$.

Unless otherwise stated, for the rest of **Section 4**, $\mathcal{D}$, together with a forgetful functor $\mathrm{For} : \mathcal{D} \to \textsc{Set}_{\mathcal{U}}$, will be a good concrete $\mathcal{U}$-category.

(b) Following our motivating example for presheaves in (**1.4.3**(b)), we use the following notation for a presheaf F in $\textsc{PSh}(X, \mathcal{D})$. For an inclusion morphism $\iota : U \to V$ in $\textsc{Open}(X)$ and $s \in \mathrm{For}FV$, as $\mathrm{For}FV$ is a set, we write $s|_U$ for the image $(\mathrm{For}(F\iota))(s) \in \mathrm{For}FU$.

(c) Since For is faithful, any morphism $f : A \to B$ in $\mathcal{D}$ can be uniquely identified with the function $\mathrm{For}(f) : \mathrm{For}A \to \mathrm{For}B, a \mapsto (\mathrm{For}(f))(a)$ in $\textsc{Set}_{\mathcal{U}}$. We will, with an abuse of notation, often refer to and define $f : A \to B$ in $\mathcal{D}$ indirectly as a function between sets in this sense and write $a \in A$ and $a \mapsto f(a)$ instead of $a \in \mathrm{For}A$ and $a \mapsto (\mathrm{For}(f))(a)$. This abuse of notation, along with (**4.1.3**), lets us identify monomorphisms, epimorphisms and isomorphisms f in $\mathcal{D}$ with monomorphisms, epimorphisms and isomorphisms $\mathrm{For}(f)$ in $\textsc{Set}_{\mathcal{U}}$.

4.1.5 Motivation (Sheaves): For the $\mathcal{U}$-topologies (X, τ) and (Y, η), we saw the following presheaf from (**1.4.3**(b)): $C : \textsc{Open}(X)^{\mathrm{op}} \to \textsc{Set}_{\mathcal{U}}$ defined through $\mathcal{U}$-sets of continuous functions $C(U, Y)$. A useful property is that for all $\mathcal{U}$-open covers $\mathfrak{U} = (U_i)_{i \in I}$ of U (i.e. $I \in \mathcal{U}$, $U = \bigcup_{i \in I} U_i$ and

$U_i \in \tau$ for all $i \in I$), a continuous function $f \in C(U,Y)$ bijects with $(f|_{U_i})_{i\in I} \in \{(s_i)_{i\in I} \in (C(U_i,Y))_{i\in I} |$ For all $i,j \in I$: $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}\}$. That is to say f *separates* injectively into its restrictions $(f|_{U_i})_{i\in I}$, and also $(f|_{U_i})_{i\in I}$ *glues* back to f uniquely from its restrictions $(f|_{U_i})_{i\in I}$, as these restrictions agree on all intersections $U_i \cap U_j$. We generalize these two notions of separating and gluing categorically to define sheaves and separated presheaves.

4.1.6 Definitions (Sheaves on Topological Spaces) (III.1.1 [Mor20]**):** Let (X,τ) be a $\mathcal{U}$-topological space. Then a presheaf F in $\mathrm{PSh}(X,\mathcal{D})$ is a *$\mathcal{D}$-valued separated presheaf on X* if for every $U \in \tau$ and every $\mathcal{U}$-open cover $\mathfrak{U} = (U_i)_{i\in I}$ of U, it follows that:

(i) *Separation*: The separation map $\iota_{U,\mathfrak{U}} : FU \to \prod_{i\in I} FU_i, s \mapsto (s|_{U_i})_{i\in I}$, induced by the morphisms $(F\iota_i : FU \to FU_i)_{i\in I}$, is a monomorphism.

Furthermore, F is a *$\mathcal{D}$-valued sheaf on X* if it is a $\mathcal{D}$-valued separated presheaf on X and if for every $U \in \tau$ and every $\mathcal{U}$-open cover $\mathfrak{U} = (U_i)_{i\in I}$ of U, we have:

(ii) *Gluing*: For the morphisms $f_{U,\mathfrak{U}}, g_{U,\mathfrak{U}} : \prod_{i\in I} FU_i \to \prod_{i,i'\in I} F(U_i \cap U_{i'})$ in $\mathcal{D}$, whereby $f_{U,\mathfrak{U}} : (s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_{i'}})_{i,i'\in I}$ and $g_{U,\mathfrak{U}} : (s_i)_{i\in I} \mapsto (s_{i'}|_{U_i\cap U_{i'}})_{i,i'\in I}$, the canonical morphism $\iota_{U,\mathfrak{U}} : FU \to \mathrm{Eq}(f_{U,\mathfrak{U}}, g_{U,\mathfrak{U}})$, induced by $\iota_{U,\mathfrak{U}}$ from (i), is an epimorphism.

4.1.7 Note (Equivalent Definitions): In (**4.1.6**), it is easy to check that (i) is also equivalent to $\iota_{U,\mathfrak{U}} : FU \to \mathrm{Eq}(f_{U,\mathfrak{U}}, g_{U,\mathfrak{U}})$ being a monomorphism. Thus (i) and (ii) together is equivalent to $\iota_{U,\mathfrak{U}} : FU \to \mathrm{Eq}(f_{U,\mathfrak{U}}, g_{U,\mathfrak{U}})$ being an isomorphism.

We often write $\mathcal{H}_U(\mathfrak{U}, F) = \mathrm{Eq}(f_{U,\mathfrak{U}}, g_{U,\mathfrak{U}})$, which has an underlying set that is equal or bijects to:

$$\left\{(s_i)_{i\in I} \in \prod_{i\in I} FU_i | \text{ For all } i,i' \in I : s_i|_{U_i\cap U_{i'}} = s_{i'}|_{U_i\cap U_{i'}}\right\}.$$

4.1.8 Definitions (Categories of Sheaves on Topological Spaces): Let (X,τ) be a $\mathcal{U}$-topological space. Then $\mathrm{Sh}(\mathrm{Open}(X),\mathcal{D}) = \mathrm{Sh}(X,\mathcal{D})$ is the *category of $\mathcal{D}$-valued sheaves on X*. Furthermore we write $\mathrm{Sh}(\mathrm{Open}(X), \mathrm{Set}_{\mathcal{U}}) = \mathrm{Sh}(\mathrm{Open}(X)) = \mathrm{Sh}(X)$, and for a $\mathcal{U}$-ring R, we often write $\mathrm{Sh}(X, {}_R\mathrm{Mod}_{\mathcal{U}}) = \mathrm{Sh}(X,R)$. These categories are full subcategories of their presheaf counterparts from (**4.1.4**(a)).

4.2 Sheaves on Grothendieck Pretopologies

We want to generalize sheaves on a topological space so that we can construct sheaves out of presheaves where the domain category is not always $\mathrm{Open}(X)$ for a $\mathcal{U}$-topology X, to do this we define Grothendieck pretopologies. Let $\mathcal{U}$ be a Grothendieck universe, let $\mathcal{C}$ be a $\mathcal{U}$-category such that $\mathcal{C}$ contains all fiber products and let $\mathcal{D}$ be a good concrete $\mathcal{U}$-category with the forgetful functor $\mathrm{For} : \mathcal{D} \to \mathrm{Set}_{\mathcal{U}}$.

4.2.1 Definition (Grothendieck Pretopologies) (III.2.1.1 [Mor20]**), (**[NCa21c]**):** A *$\mathcal{U}$-Grothendieck pretopology $\mathcal{T}$ on $\mathcal{C}$*, written $(\mathcal{C},\mathcal{T})$, contains the following information: Every object $U \in \mathrm{Ob}(\mathcal{C})$ has at least one corresponding $\mathcal{U}$-collection of morphisms $\mathfrak{U} = (u_i : U_i \to U)_{i\in I}$, $I \in \mathcal{U}$, called a *covering family of U*. These covering families altogether fulfill the following properties:

(i) *Base Changes:* If $\mathfrak{V} = (v_j : V_j \to V)_{j\in J}$ is a covering family and $f : U \to V$ is a morphism in $\mathcal{C}$, then with the fiber product $V_j \times_V U$ induced from $v_j : V_j \to V$ and change of base $f : U \to V$ for each $j \in J$, the projections $p_{U,j} : V_j \times_V U \to U$ of the fiber products form a covering family $f^*\mathfrak{V} = (p_{U,j} : V_j \times_V U \to U)_{j\in J}$ of U.

(ii) *Compositions:* If $\mathfrak{U} = (u_i : U_i \to U)_{i\in I}$ is a covering family of U and if $\mathfrak{U}_i = (u_{ij} : U_{ij} \to U_i)_{i\in J_i}$ is a covering family of U_i for all $i \in I$, then so is $\mathfrak{U}' = (u_i \circ u_{ij} : U_{ij} \to U)_{i\in I, j\in J_i}$ a covering family of U.

(iii) *Isomorphisms:* If $f : U \to V$ is an isomorphism, $\mathfrak{V} = (f : U \to V)$ is a covering family of V.

4.2.2 Definitions (Sheaves on Grothendieck Pretopologies) (III.2.1.5 [Mor20]): For a $\mathcal{U}$-Grothendieck pretopology $\mathcal{T}$ on $\mathcal{C}$, F is a *$\mathcal{D}$-valued presheaf on* $(\mathcal{C}, \mathcal{T})$ if it is an object of $\mathrm{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D}) = \mathrm{PSh}(\mathcal{C}, \mathcal{D})$ as defined in (**1.4.1**). F is called a *$\mathcal{D}$-valued separated presheaf on* $(\mathcal{C}, \mathcal{T})$ if for every covering family $\mathfrak{U} = (u_i : U_i \to U)_{i \in I}$ of every object U in $\mathcal{C}$, the following applies:

(i) *Separation*: The separation map $\iota_{U,\mathfrak{U}} : FU \to \prod_{i \in I} FU_i, s \mapsto (Fu_i(s))_{i \in I}$, induced by the morphisms $(Fu_i : FU \to FU_i)_{i \in I}$, is a monomorphism.

Furthermore, F is a *$\mathcal{D}$-valued sheaf on* $(\mathcal{C}, \mathcal{T})$ if it is a *$\mathcal{D}$-valued separated presheaf on* $(\mathcal{C}, \mathcal{T})$ and for every covering family $\mathfrak{U} = (u_i : U_i \to U)_{i \in I}$ of every object U in $\mathcal{C}$, we have also:

(ii) *Gluing*: A covering family $(u_i : U_i \to U)_{i \in I}$ induces the covering families $u_{i'}^* \mathfrak{U} = (p_{U_{i'},i} : U_i \times_U U_{i'} \to U_{i'})_{i \in I}$ of $U_{i'}$, where the change of base $u_{i'} : U_{i'} \to U$ is used, for all $i' \in I$. Then for the morphisms $f_{U,\mathfrak{U}}, g_{U,\mathfrak{U}} : \prod_{i \in I} FU_i \to \prod_{i,i' \in I} F(U_i \times_U U_{i'})$ in $\mathcal{D}$, whereby $f_{U,\mathfrak{U}}((s_i)_{i \in I}) = (Fp_{U_i,i'}(s_i))_{i,i' \in I}$ and $g_{U,\mathfrak{U}}((s_i)_{i \in I}) = (Fp_{U_{i'},i}(s_{i'}))_{i,i' \in I}$, the canonical morphism $\iota_{U,\mathfrak{U}} : FU \to \mathrm{Eq}(f_{U,\mathfrak{U}}, g_{U,\mathfrak{U}})$, induced by $\iota_{U,\mathfrak{U}}$ from (i), is an epimorphism.

4.2.3 Note (Equivalent Definitions) (III.2.2.4 [Mor20]): In (**4.2.2**), it is easy to check that (i) is also equivalent to $\iota_{U,\mathfrak{U}} : FU \to \mathrm{Eq}(f_{U,\mathfrak{U}}, g_{U,\mathfrak{U}})$ being a monomorphism. Thus (i) and (ii) together is equivalent to $\iota_{U,\mathfrak{U}} : FU \to \mathrm{Eq}(f_{U,\mathfrak{U}}, g_{U,\mathfrak{U}})$ being an isomorphism.

We often write $\mathcal{H}_U(\mathfrak{U}, F) = \mathrm{Eq}(f_{U,\mathfrak{U}}, g_{U,\mathfrak{U}})$, which has an underlying set that is equal or bijects to:

$$\left\{ (s_i)_{i \in I} \in \prod_{i \in I} FU_i | \text{ For all } i, i' \in I : (Fp_{U_i,i'}(s_i))_{i,i' \in I} = (Fp_{U_{i'},i}(s_{i'}))_{i,i' \in I} \right\}.$$

4.2.4 Definitions (Categories of Sheaves on Grothendieck Pretopologies): For a $\mathcal{U}$-Grothendieck pretopology $(\mathcal{C}, \mathcal{T})$, we denote $\mathrm{Sh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ as the *category of $\mathcal{D}$-valued sheaves on* $(\mathcal{C}, \mathcal{T})$ as a full subcategory of $\mathrm{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$. We also write $\mathrm{Sh}((\mathcal{C}, \mathcal{T}), \mathrm{Set}_{\mathcal{U}}) = \mathrm{Sh}((\mathcal{C}, \mathcal{T}))$. For a $\mathcal{U}$-ring R, we also write $\mathrm{Sh}((\mathcal{C}, \mathcal{T}), {}_R\mathrm{Mod}_{\mathcal{U}}) = \mathrm{Sh}((\mathcal{C}, \mathcal{T}), R)$, which is a full subcategory of $\mathrm{PSh}((\mathcal{C}, \mathcal{T}), R)$.

4.2.5 Note (Grothendieck Pretopologies): As stated before, $\mathcal{U}$-Grothendieck pretopologies are a generalization of the categories $\mathrm{Open}(X)$ for a $\mathcal{U}$-topological space (X, τ). $\mathrm{Open}(X)$ has a canonical $\mathcal{U}$-Grothendieck pretopology structure $\mathcal{T}$, whereby the covering families of an object U in $\mathrm{Open}(X)$ are precisely the inclusion morphisms $\mathfrak{U} = (\iota_i : U_i \to U)_{i \in I}$ whereby $(U_i)_{i \in I}$ is a $\mathcal{U}$-open cover of U. Then it is clear that for all $i, i' \in I$, we have $U_i \times_U U_{i'} = U_i \cap U_{i'}$ in terms of $\mathcal{U}$-sets. Furthermore, the two categories $\mathrm{Sh}(X, \mathcal{D})$ and $\mathrm{Sh}((\mathrm{Open}(X), \mathcal{T}), \mathcal{D})$ have a canonical category-isomorphism to each other.

4.2.6 Definitions (Categories of Covering Families):

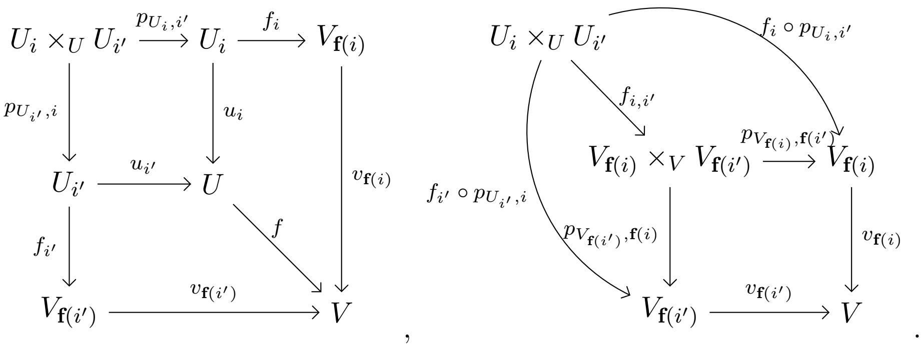
(a) **Categories of Covering Families (III.2.2.1 [Mor20]):** Let $(\mathcal{C}, \mathcal{T})$ be a $\mathcal{U}$-Grothendieck pretopology and let U and V be objects in $\mathcal{C}$. Let $\mathfrak{U} = (u_i : U_i \to U)_{i \in I}$ and $\mathfrak{V} = (v_j : V_j \to V)_{j \in J}$ be covering families of U and V respectively. A *morphism of covering families* $f : \mathfrak{U} \to \mathfrak{V}$ consists of an index mapping $\mathbf{f} : I \to J$, a morphism $f : U \to V$ in $\mathcal{C}$ and a collection of morphisms $(f_i : U_i \to V_{\mathbf{f}(i)})_{i \in I}$ in $\mathcal{C}$, such that for every $i \in I$, the following diagram commutes:

$$\begin{array}{ccc} U_i & \xrightarrow{f_i} & V_{\mathbf{f}(i)} \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle v_{\mathbf{f}(i)}} \\ U & \xrightarrow{f} & V \end{array}.$$

A simple calculation with commutative diagrams shows that the compositions of two morphisms of covering families $f : \mathfrak{U} \to \mathfrak{V}$ and $g : \mathfrak{V} \to \mathfrak{W}$, given with the index mapping $\mathbf{g} \circ \mathbf{f}$, morphism $g \circ f : U \to W$ and family of morphisms $(g_{\mathbf{f}(i)} \circ f_i : U_i \to W_{\mathbf{g} \circ \mathbf{f}(i)})_{i \in I}$, is a morphism of covering families. These morphisms, along with the covering families as objects, form the *$\mathcal{U}$-category of covering families in* $(\mathcal{C}, \mathcal{T})$, denoted by $\mathrm{Cov}(\mathcal{C}, \mathcal{T})$. $\mathrm{Cov}(\mathcal{C}, \mathcal{T})$ is a $\mathcal{U}$-category, as collections of morphisms between covering families $\mathfrak{U}$ of U and $\mathfrak{V}$ of V biject to subsets of $\coprod_{f\in \mathrm{Hom}_{\mathcal{C}}(U,V), \mathbf{f}\in \mathrm{Hom}_{\mathrm{SET}_{\mathcal{U}}}(I,J)} \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(U_i, V_{\mathbf{f}(i)})$, which is a $\mathcal{U}$-set due to (**I.1.3**(v) [Mor20]) and (**I.1.3**(viii) [Mor20]).

(b) **Induced Morphisms on $\mathcal{H}_U(\mathfrak{U}, F)$ and $\mathcal{H}_V(\mathfrak{V}, F)$ (III.2.2.4** [Mor20]**):** For a $\mathcal{U}$-Grothendieck pretopology $(\mathcal{C}, \mathcal{T})$, let F be a presheaf in $\mathrm{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$. For objects U and V in $\mathcal{C}$ with the covering families $\mathfrak{U} = (u_i : U_i \to U)_{i\in I}$ and $\mathfrak{V} = (v_j : V_j \to V)_{j\in J}$, let $f : \mathfrak{U} \to \mathfrak{V}$ be a morphism in $\mathrm{Cov}(\mathcal{C}, \mathcal{T})$. f consists of the information $\mathbf{f} : I \to J$, $f : U \to V$ in $\mathcal{C}$ and $(f_i : U_i \to V_{\mathbf{f}(i)})_{i\in I}$ in $\mathcal{C}$. We then induce a mapping $f_{\mathcal{H}} : \mathcal{H}_V(\mathfrak{V}, F) \to \mathcal{H}_U(\mathfrak{U}, F)$ through $(s_j)_{j\in J} \in \mathcal{H}_V(\mathfrak{V}, F) \subset \prod_{j\in J} FV_j$ with the mapping $(s_j)_{j\in J} \mapsto (Ff_i(s_{\mathbf{f}(i)}))_{i\in I} \in \prod_{i\in I} FU_i$.

We claim that $f_{\mathcal{H}}$ is well-defined, i.e. that $(Ff_i(s_{\mathbf{f}(i)}))_{i\in I} \in \mathcal{H}_U(\mathfrak{U}, F)$. Let $u_{i'}^*\mathfrak{U} = (p_{U_{i'},i} : U_i \times_U U_{i'} \to U_{i'})_{i\in I}$ be a covering family for $U_{i'}$ for all $i' \in I$, where the change of base $u_{i'} : U_{i'} \to U$ is used. Let $v_{j'}^*\mathfrak{V} = (p_{V_{j'},j} : V_i \times_V V_{j'} \to V_{j'})_{j\in I}$ be a covering family for $V_{j'}$ for all $j' \in J$, where the change of base $v_{j'} : V_{j'} \to V$ is used. Observe that the commutativity of the left diagram implies the existence of $f_{i,i'}$ in the right diagram, which also commutes:



Applying F to the above diagrams, and thus inverting the arrows, gives us:

$$Fp_{U_i,i'}(Ff_i(s_{\mathbf{f}(i)})) = Ff_{i,i'}(Fp_{V_{\mathbf{f}(i)},\mathbf{f}(i')}(s_{\mathbf{f}(i)})),$$

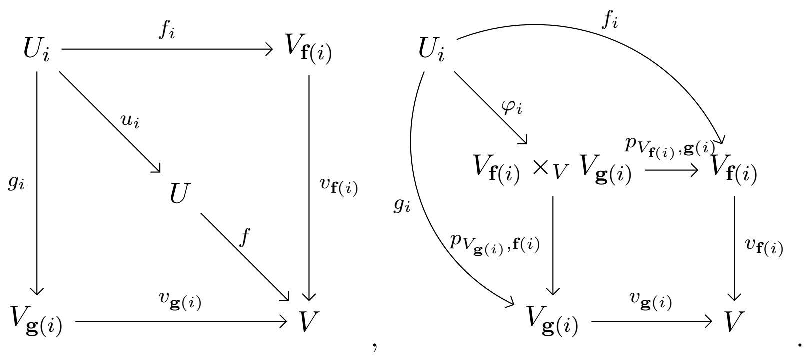
Due to $(s_j)_{j\in J} \in \mathcal{H}_V(\mathfrak{V}, F)$ we have that $Fp_{V_{\mathbf{f}(i)},\mathbf{f}(i')}(s_{\mathbf{f}(i)}) = Fp_{V_{\mathbf{f}(i')},\mathbf{f}(i)}(s_{\mathbf{f}(i')})$, thus:

$$\begin{aligned} &= Ff_{i,i'}(Fp_{V_{\mathbf{f}(i')},\mathbf{f}(i)}(s_{\mathbf{f}(i')}))), \\ &= Fp_{U_{i'},i}(Ff_{i'}(s_{\mathbf{f}(i')})). \end{aligned}$$

Thus we have $(Ff_i(s_{\mathbf{f}(i)}))_{i\in I} \in \mathcal{H}_U(\mathfrak{U}, F)$ and that $f_{\mathcal{H}}$ is a well-defined morphism in $\mathcal{D}$.

4.2.7 Lemma (Covering Families) (III.2.2.7 [Mor20]**):** Let $(\mathcal{C}, \mathcal{T})$ be a $\mathcal{U}$-Grothendieck pretopology and F be a presheaf in $\mathrm{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$. For the objects U and V in $\mathcal{C}$, let $f, g : \mathfrak{U} \to \mathfrak{V}$ be morphisms in $\mathrm{Cov}(\mathcal{C}, \mathcal{T})$ for the covering families $\mathfrak{U} = (u_i : U_i \to U)_{i\in I}$ and $\mathfrak{V} = (v_j : V_j \to V)_{j\in J}$, such that f and g have the following information and conditions: $\mathbf{f}, \mathbf{g} : I \to J$, $f = g : U \to V$ in $\mathcal{C}$ and $(f_i, g_i : U_i \to V_{\mathbf{f}(i)})_{i\in I}$ in $\mathcal{C}$. Then $f_{\mathcal{H}} = g_{\mathcal{H}} : \mathcal{H}_V(\mathfrak{V}, F) \to \mathcal{H}_U(\mathfrak{U}, F)$ as morphisms in $\mathcal{D}$.

Proof: We want to show that for any $(s_j)_{j\in J} \in \mathcal{H}_V(\mathfrak{V}, F) \subset \prod_{j\in J} FV_j$ we have $Ff_i(s_{\mathbf{f}(i)}) = Fg_i(s_{\mathbf{g}(i)})$ for all $i \in I$. Let $i \in I$, since the first diagram below commutes, φ_i exists in the second diagram below such that the second diagram commutes:



Applying F to the above diagrams, and thus inverting the arrows, gives us:

$$Ff_i(s_{\mathbf{f}(i)}) = F\varphi_i(Fp_{V_{\mathbf{f}(i)},\mathbf{g}(i)}(s_{\mathbf{f}(i)})),$$

Due to $(s_j)_{j\in J} \in \mathcal{H}_V(\mathfrak{V}, F)$, we have that $Fp_{V_{\mathbf{f}(i)},\mathbf{g}(i)}(s_{\mathbf{f}(i)}) = Fp_{V_{\mathbf{g}(i)},\mathbf{f}(i)}(s_{\mathbf{g}(i)})$, thus:

$$\begin{aligned} &= F\varphi_i(Fp_{V_{\mathbf{g}(i)},\mathbf{f}(i)}(s_{\mathbf{g}(i)})), \\ &= Fg_i(s_{\mathbf{g}(i)}). \end{aligned}$$

Thus the claim follows. □

4.2.8 Definition (Categories of Covering Families of U) (III.2.2.2 [Mor20]**):** Let $(\mathcal{C}, \mathcal{T})$ be a $\mathcal{U}$-Grothendieck pretopology with an object U in $\mathcal{C}$. The *$\mathcal{U}$-category of covering families of U* $\mathrm{Cov}(U)$ is the category where covering families of U, $\mathfrak{U} = (u_i : U_i \to U)_{i\in I}$, are the objects. A morphism in $\mathrm{Cov}(U)$ between two covering families $\mathfrak{U} = (u_i : U_i \to U)_{i\in I}$ and $\mathfrak{U}' = (u'_{i'} : U'_{i'} \to U)_{i'\in I'}$ is given as a morphism of covering families $f : \mathfrak{U} \to \mathfrak{U}'$, as seen in (**4.2.6**(a)), with the extra condition that the corresponding morphism $f : U \to U$ in $\mathcal{C}$ is the identity 1_U. It is clear to see that composition works and that $\mathrm{Cov}(U)$ is a $\mathcal{U}$-subcategory of $\mathrm{Cov}(\mathcal{C}, \mathcal{T})$.

4.2.9 Definition (Reduced Categories of Covering Families of U) (III.2.2.2 [Mor20]**):** For a $\mathcal{U}$-Grothendieck pretopology $(\mathcal{C}, \mathcal{T})$ and an object U in $\mathcal{C}$, we define the category $\mathrm{Cov}^0(U)$ as the *reduced $\mathcal{U}$-category of covering families of U*. $\mathrm{Cov}^0(U)$ contains the same objects as $\mathrm{Cov}(U)$, i.e. covering families of U, however, for any two objects $\mathfrak{U}$ and $\mathfrak{U}'$ in $\mathrm{Cov}^0(U)$, there exists at most one morphism $\varphi_{\mathfrak{U},\mathfrak{U}'} : \mathfrak{U} \to \mathfrak{U}'$ in $\mathrm{Cov}^0(U)$. If $\mathrm{Hom}_{\mathrm{Cov}(U)}(\mathfrak{U}, \mathfrak{U}')$ is empty, then $\mathrm{Hom}_{\mathrm{Cov}^0(U)}(\mathfrak{U}, \mathfrak{U}')$ is empty, if $\mathrm{Hom}_{\mathrm{Cov}(U)}(\mathfrak{U}, \mathfrak{U}')$ is nonempty, then $\mathrm{Hom}_{\mathrm{Cov}^0(U)}(\mathfrak{U}, \mathfrak{U}')$ contains exactly one morphism, labeled $\varphi_{\mathfrak{U},\mathfrak{U}'} : \mathfrak{U} \to \mathfrak{U}'$. It is easy to calculate that $\mathrm{Cov}^0(U)$ defines a $\mathcal{U}$-category. If such a morphism $\varphi_{\mathfrak{U},\mathfrak{U}'}$ exists, we say that $\mathfrak{U}$ is a *refinement* of $\mathfrak{U}'$.

4.2.10 Notes (Reduced Categories of Covering Families of U) (III.2.2.7 [Mor20]**):** Let $(\mathcal{C}, \mathcal{T})$ be a $\mathcal{U}$-Grothendieck pretopology and U be an object in $\mathcal{C}$.

(a) There exists a canonical functor $\varphi : \mathrm{Cov}(U) \to \mathrm{Cov}^0(U)$ which maps objects to themselves and sends any morphism $f : \mathfrak{U} \to \mathfrak{U}'$ in $\mathrm{Cov}(U)$ to the unique morphism $\varphi_{\mathfrak{U},\mathfrak{U}'} : \mathfrak{U} \to \mathfrak{U}'$ in $\mathrm{Cov}^0(U)$.

(b) $\mathrm{Cov}^0(U)^{\mathrm{op}}$ is a filtered $\mathcal{U}$-category, to show this we prove the conditions in (**1.5.14**):

For (**1.5.14**(i)): $\mathrm{Cov}^0(U)^{\mathrm{op}}$ is not empty as U has the trivial covering family $\mathfrak{U} = (1_U : U \to U)$.

For (**1.5.14**(ii)): For any two covering families $\mathfrak{U} = (u_i : U_i \to U)_{i\in I}$ and $\mathfrak{U}' = (u'_{i'} : U'_{i'} \to U)_{i'\in I'}$ of U, observe the change of base $u'_{i'} : U'_{i'} \to U$ on $\mathfrak{U}$ for all $i' \in I'$, i.e. $u'^*_{i'}\mathfrak{U} = (p_{U'_{i'},i} : U_i \times_U U'_{i'} \to U'_{i'})_{i\in I}$. $u'^*_{i'}\mathfrak{U}$ is a covering family of $U'_{i'}$ for all $i' \in I'$. Then compose the covering families $(u'^*_{i'}\mathfrak{U})_{i'\in I'}$ with $\mathfrak{U}' = (u'_{i'} : U'_{i'} \to U)_{i'\in I'}$ to get a new covering family $\mathfrak{V} = (\psi_{i,i'} : U_i \times_U U'_{i'} \to U)_{i \in I, i' \in I'}$ whereby $\psi_{i,i'} = u'_{i'} \circ p_{U'_{i'},i} = u_i \circ p_{U_i,i'}$. It is clear that the morphism $p_{\mathfrak{U}} = 1_U : U \to U$, with the projection $\mathbf{p}_{\mathfrak{U}} : I \times I' \to I$ onto I and the projection morphisms $(p_{U_i,i'} : U_i \times_U U'_{i'} \to U_i)_{i \in I, i' \in I'}$, defines a morphism $p_{\mathfrak{U}} : \mathfrak{V} \to \mathfrak{U}$ in $\text{Cov}(U)$. Analogously, we define $p_{\mathfrak{U}'} : \mathfrak{V} \to \mathfrak{U}'$, which gives us morphisms $(p_U)^{\text{op}} : \mathfrak{U} \to \mathfrak{V}$ and $(p_V)^{\text{op}} : \mathfrak{U}' \to \mathfrak{V}$ in $\text{Cov}^0(U)^{\text{op}}$ as claimed.

For (**1.5.14**(iii)): This is obviously fulfilled due to $\text{Hom}_{\text{Cov}^0(U)^{\text{op}}}(\mathfrak{U}, \mathfrak{U}')$ being either a singleton or empty for all open covers $\mathfrak{U}$ and $\mathfrak{U}'$.

(c) For a presheaf F in $\text{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$, there exists a functor $\mathcal{H}_U(\_, F) : \text{Cov}(U)^{\text{op}} \to \mathcal{D}$ whereby $\mathcal{H}_U(\_, F)$ maps covering families $\mathfrak{U}$ to $\mathcal{H}_U(\mathfrak{U}, F)$. $\mathcal{H}_U(\_, F)$ maps morphisms of covering families $f : \mathfrak{U} \to \mathfrak{U}'$ in $\text{Cov}(U)$ to $f_{\mathcal{H}} : \mathcal{H}_U(\mathfrak{U}', F) \to \mathcal{H}_U(\mathfrak{U}, F)$, as seen in (**4.2.6**(b)). It is easy to check that $\mathcal{H}_U(\_, F)$ is a functor due to the functoriality of F, and the definition of morphisms of covering families.

Due to (**4.2.7**), for any two morphisms $f, g : \mathfrak{U} \to \mathfrak{U}'$ in $\text{Cov}(U)$, we have that $f_{\mathcal{H}} = g_{\mathcal{H}}$. Thus, $\mathcal{H}_U(\_, F)$ factorizes through $\text{Cov}^0(U)$, i.e. $\mathcal{H}_U(\_, F) : \text{Cov}^0(U)^{\text{op}} \to \mathcal{D}$, defined with the mappings $\mathfrak{U} \mapsto \mathcal{H}_U(\mathfrak{U}, F)$ and $(\varphi_{\mathfrak{U},\mathfrak{U}'})^{\text{op}} \mapsto f_{\mathcal{H}}$ for any $f : \mathfrak{U} \to \mathfrak{U}'$ in $\text{Cov}(U)$, is well-defined and functorial. Furthermore, $\mathcal{H}_U(\_, F) : \text{Cov}(U)^{\text{op}} \to \mathcal{D}$ is the same functor as $\varphi^{\text{op}} : \text{Cov}(U)^{\text{op}} \to \text{Cov}^0(U)^{\text{op}}$ from (**4.2.10**(a)) in composition with $\mathcal{H}_U(\_, F) : \text{Cov}^0(U)^{\text{op}} \to \mathcal{D}$. In future contexts, $\mathcal{H}_U(\_, F)$ will denote the functor from $\text{Cov}^0(U)^{\text{op}}$.

4.2.11 Note ($\mathcal{U}$-Small Covering Families): In order to define separation and sheafification functors, we will not only require a $\mathcal{U}$-Grothendieck pretopology $(\mathcal{C}, \mathcal{T})$, but also require that for all objects U in $\mathcal{C}$, the categories $\text{Cov}(U)$ and $\text{Cov}^0(U)$ are $\mathcal{U}$-small. We denote this property of $(\mathcal{C}, \mathcal{T})$ as $(\mathcal{C}, \mathcal{T})$ being a *$\mathcal{U}$-Grothendieck pretopology with $\mathcal{U}$-small covering families.*

Our previous example of a $\mathcal{U}$-Grothendieck pretopology in (**4.2.5**) clearly fulfills this condition.

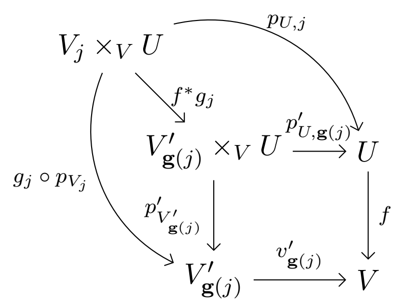
From now on, $(\mathcal{C}, \mathcal{T})$ will always denote a $\mathcal{U}$-Grothendieck pretopology with $\mathcal{U}$-small covering families.

4.2.12 Definition (Separation Functors) (III.2.2.6, III.2.2.9 [Mor20]**):** We define a functor $(\_)^{\text{sep}} : \text{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D}) \to \text{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ as follows:

For object mapping of $(\_)^{\text{sep}}$: Let F be a presheaf in $\text{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ and for every object U in $\mathcal{C}$, let $F^{\text{sep}}U = \text{colim}_{\text{Cov}^0(U)^{\text{op}}} \mathcal{H}_U(\_, F)$ be an object in $\mathcal{D}$. This colimit exists as $\mathcal{D}$ is a good concrete $\mathcal{U}$-category and thus $\mathcal{D}$ has all $\mathcal{U}$-small filtered colimits.

For any morphism $f : U \to V$ in $\mathcal{C}$, we have a functor $f_{\text{Cov}} : \text{Cov}^0(V) \to \text{Cov}^0(U)$ which maps objects $\mathfrak{V} \mapsto f^*\mathfrak{V}$ as the change of base induced by f, f_{Cov} maps morphisms $\varphi_{\mathfrak{V},\mathfrak{V}'}$ in $\text{Cov}^0(V)$ to $\varphi_{f^*\mathfrak{V}, f^*\mathfrak{V}'}$ in $\text{Cov}^0(U)$. With $\mathfrak{V} = (v_j : V_j \to V)_{j \in J}$ and $\mathfrak{V}' = (v'_{j'} : V'_{j'} \to V)_{j' \in J'}$, the existence of $\varphi_{f^*\mathfrak{V}, f^*\mathfrak{V}'}$ is proven as follows: The existence of $\varphi_{\mathfrak{V},\mathfrak{V}'}$ implies that there exists a morphism $g : \mathfrak{V} \to \mathfrak{V}'$ in $\text{Cov}(V)$ with morphisms $\mathbf{g} : J \to J'$ and $(g_j : V_j \to V'_{\mathbf{g}(j)})_{j \in J}$. It is easy to check that for all $j \in J$, the fiber product $(V_j \times_V U, p_{U,j}, p_{V_j})$ induced from f and v_j and the fiber product $(V'_{\mathbf{g}(j)} \times_V U, p'_{U,\mathbf{g}(j)}, p'_{V'_{\mathbf{g}(j)}})$ induced from f and $v'_{\mathbf{g}(j)}$ induces a morphism $f^*g_j : V_j \times_V U \to V'_{\mathbf{g}(j)} \times_V U$, since $v'_{\mathbf{g}(j)} \circ g_j = v_j$ and since the following diagram commutes:

($\star$)



It is easy to check that $f^*g : f^*\mathfrak{V} \to f^*\mathfrak{V}'$, given by $\mathbf{f^*g} = \mathbf{g} : J \to J'$ and the morphisms $(f^*g_j : V_j \times_V U \to V'_{\mathbf{g}(j)} \times_V U)_{j \in J}$, forms a morphism in $\text{Cov}(U)$, implying that $\varphi_{f^*\mathfrak{V}, f^*\mathfrak{V}'}$ exists.

The functoriality of f_{Cov} is also clear. Thus, we have a functor $\mathcal{H}_U(\_, F) \circ f^{\mathrm{op}}_{\mathrm{Cov}} : \mathrm{Cov}^0(V)^{\mathrm{op}} \to \mathcal{D}$ that maps objects $\mathfrak{V} \to \mathcal{H}_U(f^*\mathfrak{V}, F)$ and morphisms $\varphi_{\mathfrak{V},\mathfrak{V}'} \mapsto g_{\mathcal{H}}$, for any $g : f^*\mathfrak{V} \to f^*\mathfrak{V}'$ in $\mathrm{Cov}(U)$.

For any covering family $\mathfrak{V} = (v_j : V_j \to V)_{j\in J}$ in $\mathrm{Cov}^0(V)$, we have the canonical morphism of covering families $f_{\mathfrak{V}} : f^*\mathfrak{V} \to \mathfrak{V}$ in $\mathrm{Cov}(\mathcal{C}, \mathcal{T})$ given by $f : U \to V$, the identity $\mathbf{f} = \mathrm{id}_j : J \to J$ and the projection morphisms $(p_{V_j} : V_j \times_V U \to V_j)_{j\in J}$. This morphism $f_{\mathfrak{V}} : f^*\mathfrak{V} \to \mathfrak{V}$ induces a morphism $f_{\mathfrak{V}\mathcal{H}} : \mathcal{H}_V(\mathfrak{V}, F) \to \mathcal{H}_U(f^*\mathfrak{V}, F), (s_j)_{j\in J} \mapsto (Fp_{V_j}(s_j))_{j\in J}$ as defined in (**4.2.6**(b)). We claim that these morphisms $(f_{\mathfrak{V}\mathcal{H}})_{\mathfrak{V}\in \mathrm{Ob}(\mathrm{Cov}^0(V)^{\mathrm{op}})}$ induce a natural transformation $u_f : \mathcal{H}_V(\_, F) \to \mathcal{H}_U(\_, F) \circ f^{\mathrm{op}}_{\mathrm{Cov}}$ and to prove this, it is enough to show that the following diagram commutes for all morphisms $g : \mathfrak{V} \to \mathfrak{V}'$ in $\mathrm{Cov}(V)$:

$$\begin{array}{ccc} \mathcal{H}_V(\mathfrak{V}', F) & \xrightarrow{f_{\mathfrak{V}'\mathcal{H}}} & \mathcal{H}_U(f^*\mathfrak{V}', F) \\ {\scriptstyle g_{\mathcal{H}}}\downarrow & & \downarrow{\scriptstyle (f^*g)_{\mathcal{H}}} \\ \mathcal{H}_V(\mathfrak{V}, F) & \xrightarrow{f_{\mathfrak{V}\mathcal{H}}} & \mathcal{H}_U(f^*\mathfrak{V}, F) \end{array} .$$

With $\mathfrak{V} = (v_j : V_j \to V)_{j\in J}$ and $\mathfrak{V} = (v'_{j'} : V'_{j'} \to V)_{j'\in J'}$ and the projection morphisms $(p_{V_j} : V_j \times_V U \to V_j)_{j\in J}$ and $(p'_{V'_{j'}} : V'_{j'} \times_V U \to V'_{j'})_{j'\in J'}$ we have for all $(s_{j'})_{j'\in J'} \in \mathcal{H}_V(\mathfrak{V}', F)$:

$$\begin{aligned} (f^*g)_{\mathcal{H}} \circ f_{\mathfrak{V}'\mathcal{H}}((s_{j'})_{j'\in J'}) &= (f^*g)_{\mathcal{H}}(Fp'_{V'_{j'}}(s_{j'})_{j'\in J'}), \\ &= ((Ff^*g_j Fp'_{V'_{\mathbf{g}(j)}})(s_{\mathbf{g}(j)}))_{j\in J}, \end{aligned}$$

The fiber product diagram $(\star)$ with F applied, gives us that:

$$\begin{aligned} &= ((Fp_{V_j} Fg_j)(s_{\mathbf{g}(j)}))_{j\in J}, \\ &= f_{\mathfrak{V}\mathcal{H}} \circ g_{\mathcal{H}}((s_{j'})_{j'\in J'}). \end{aligned}$$

This natural transformation $u_f : \mathcal{H}_V(\_, F) \to \mathcal{H}_U(\_, F) \circ f^{\mathrm{op}}_{\mathrm{Cov}}$ then induces a morphism $\mathrm{colim}_{\mathrm{Cov}^0(V)^{\mathrm{op}}} u_f : \mathrm{colim}_{\mathrm{Cov}^0(V)^{\mathrm{op}}} \mathcal{H}_V(\_, F) \to \mathrm{colim}_{\mathrm{Cov}^0(V)^{\mathrm{op}}} \mathcal{H}_U(\_, F) \circ f^{\mathrm{op}}_{\mathrm{Cov}}$ due to the colimit being a functor due to (**1.5.7**). Furthermore, the colimit cocone associated with $\mathrm{colim}_{\mathrm{Cov}^0(U)^{\mathrm{op}}} \mathcal{H}_U(\_, F)$ contains a cocone from $\mathcal{H}_U(\_, F) \circ f^{\mathrm{op}}_{\mathrm{Cov}}$ to $\mathrm{colim}_{\mathrm{Cov}^0(U)^{\mathrm{op}}} \mathcal{H}_U(\_, F)$, this is due to the fact that the image of $\mathcal{H}_U(\_, F) \circ f^{\mathrm{op}}_{\mathrm{Cov}}$ is a subgraph of the image of $\mathcal{H}_U(\_, F)$. Due to the colimit property, this induces a canonical morphism $v_f : \mathrm{colim}_{\mathrm{Cov}^0(V)^{\mathrm{op}}} \mathcal{H}_U(\_, F) \circ f^{\mathrm{op}}_{\mathrm{Cov}} \to \mathrm{colim}_{\mathrm{Cov}^0(U)^{\mathrm{op}}} \mathcal{H}_U(\_, F)$. We define $F^{\mathrm{sep}} f : F^{\mathrm{sep}} V \to F^{\mathrm{sep}} U$ as $v_f \circ \mathrm{colim}_{\mathrm{Cov}^0(V)^{\mathrm{op}}} u_f$.

We claim that F^{sep} defines a presheaf in $\mathrm{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$. Firstly, F^{sep} maps identities 1_A in $(\mathcal{C}, \mathcal{T})$ to themselves, as u_{1_A} and v_{1_A} in the above construction must be identities. Secondly, for two morphisms $f : U \to V$, $g : V \to W$ in $\mathcal{C}$, it can be checked that $(g \circ f)_{\mathrm{Cov}} = f_{\mathrm{Cov}} g_{\mathrm{Cov}}$ and that for all covering families $\mathfrak{W}$ in W, we have that $f_{g^*\mathfrak{W}\mathcal{H}} \circ g_{\mathfrak{W}\mathcal{H}} : \mathcal{H}_W(\mathfrak{W}, F) \to \mathcal{H}_U(f^*(g^*\mathfrak{W}), F)$, is the same as $(g \circ f)_{\mathfrak{W}\mathcal{H}} : \mathcal{H}_W(\mathfrak{W}, F) \to \mathcal{H}_U((g \circ f)^*\mathfrak{W}, F)$. Then using the functoriality of colimits, that colimits commute with each other, as seen in (**I.5.4.1** [Mor20]), and the constructions of u_f, u_g, $u_{g\circ f}$ and v_g, we have that:

$$\mathrm{colim}_{\mathrm{Cov}^0(V)^{\mathrm{op}}} u_f \circ v_g \circ \mathrm{colim}_{\mathrm{Cov}^0(W)^{\mathrm{op}}} u_g : F^{\mathrm{sep}} W \to \mathrm{colim}_{\mathrm{Cov}^0(V)^{\mathrm{op}}} \mathcal{H}_U(\_, F) \circ f^{\mathrm{op}}_{\mathrm{Cov}},$$

and $\mathrm{colim}_{\mathrm{Cov}^0(W)^{\mathrm{op}}} u_{g\circ f} : F^{\mathrm{sep}} W \to \mathrm{colim}_{\mathrm{Cov}^0(W)^{\mathrm{op}}} \mathcal{H}_U(\_, F) \circ (f^{\mathrm{op}}_{\mathrm{Cov}} g^{\mathrm{op}}_{\mathrm{Cov}})$, composed with the colimit morphism $\mathrm{colim}_{\mathrm{Cov}^0(W)^{\mathrm{op}}} \mathcal{H}_U(\_, F) \circ (f^{\mathrm{op}}_{\mathrm{Cov}} g^{\mathrm{op}}_{\mathrm{Cov}}) \to \mathrm{colim}_{\mathrm{Cov}^0(V)^{\mathrm{op}}} \mathcal{H}_U(\_, F) \circ f^{\mathrm{op}}_{\mathrm{Cov}}$, are the same morphism. Essentially v_g, defined through colimits, in the former morphism has been shifted to the end in the latter morphism, which does not affect anything. Applying $v_f : \mathrm{colim}_{\mathrm{Cov}^0(V)^{\mathrm{op}}} \mathcal{H}_U(\_, F) \circ f^{\mathrm{op}}_{\mathrm{Cov}} \to F^{\mathrm{sep}} U$ to both morphisms implies that $F^{\mathrm{sep}}(g \circ f) = F^{\mathrm{sep}} f \circ F^{\mathrm{sep}} g$.

(⋆⋆) For morphism mapping of $(\_)^{\text{sep}}$: Let $u : F \to G$ be a natural transformation in $\text{PSH}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ given by the morphisms $(uU : FU \to GU)_{U \in \text{Ob}(\mathcal{C})}$. For all objects U in $\mathcal{C}$ and all covering families $\mathfrak{U}$ of U, it can be shown that uU induces a morphism $uU_{\mathfrak{U}\mathcal{H}} : \mathcal{H}_U(\mathfrak{U}, F) \to \mathcal{H}_U(\mathfrak{U}, G), (s_i)_{i \in I} \mapsto (uU_i(s_i))_{i \in I}$. These morphisms induce a natural transformation $uU_{\_\mathcal{H}} : \mathcal{H}_U(\_, F) \to \mathcal{H}_U(\_, G)$. We then define $u^{\text{sep}}U = \text{colim}_{\text{Cov}^0(U)^{\text{op}}} uU_{\_\mathcal{H}} : F^{\text{sep}}U \to G^{\text{sep}}U$. For a morphism $f : U \to V$ in $\mathcal{C}$, it is clear that the left diagram below must commute for all covering families $\mathfrak{V}$ of V, implying the commutativity of the right diagram below:

$$\begin{array}{ccc} \mathcal{H}_V(\mathfrak{V}, F) & \xrightarrow{f_{\mathfrak{V}\mathcal{H}}} & \mathcal{H}_U(f^*\mathfrak{V}, F) \\ {\scriptstyle uV_{\mathfrak{V}\mathcal{H}}}\downarrow & & \downarrow{\scriptstyle uU_{f^*\mathfrak{V}\mathcal{H}}} \\ \mathcal{H}_V(\mathfrak{V}, G) & \xrightarrow{f_{\mathfrak{V}\mathcal{H}}} & \mathcal{H}_U(f^*\mathfrak{V}, G) \end{array}, \qquad \begin{array}{ccc} \mathcal{H}_V(\_, F) & \xrightarrow{u_f} & \mathcal{H}_U(\_, F) \circ f_{\text{Cov}}^{\text{op}} \\ {\scriptstyle uV_{\_\mathcal{H}}}\downarrow & & \downarrow{\scriptstyle uU_{\_\mathcal{H}} \circ f_{\text{Cov}}^{\text{op}}} \\ \mathcal{H}_V(\_, G) & \xrightarrow{u_f} & \mathcal{H}_U(\_, G) \circ f_{\text{Cov}}^{\text{op}} \end{array}.$$

Applying $\text{colim}_{\text{Cov}^0(V)^{\text{op}}}$ onto the right diagram above gives the commutativity of the left diagram below, which implies the commutativity of the right diagram below. This is because the horizontal morphisms on the left diagram below can be extended into $F^{\text{sep}}f$ and $G^{\text{sep}}f$ through the help of the morphisms v_f and the universal properties of colimits:

$$\begin{array}{ccc} F^{\text{sep}}V & \xrightarrow{\text{colim}_{\text{Cov}^0(V)^{\text{op}}} u_f} & \text{colim}_{\text{Cov}^0(V)^{\text{op}}} \mathcal{H}_U(\_, F) \circ f_{\text{Cov}}^{\text{op}} \\ {\scriptstyle u^{\text{sep}}V}\downarrow & & \downarrow{\scriptstyle \text{colim}_{\text{Cov}^0(V)^{\text{op}}} uU_{\_\mathcal{H}} \circ f_{\text{Cov}}^{\text{op}}} \\ G^{\text{sep}}V & \xrightarrow{\text{colim}_{\text{Cov}^0(V)^{\text{op}}} u_f} & \text{colim}_{\text{Cov}^0(V)^{\text{op}}} \mathcal{H}_U(\_, G) \circ f_{\text{Cov}}^{\text{op}} \end{array}, \qquad \begin{array}{ccc} F^{\text{sep}}V & \xrightarrow{F^{\text{sep}}f} & F^{\text{sep}}U \\ {\scriptstyle u^{\text{sep}}V}\downarrow & & \downarrow{\scriptstyle u^{\text{sep}}U} \\ G^{\text{sep}}V & \xrightarrow{G^{\text{sep}}f} & G^{\text{sep}}U \end{array}.$$

Therefore, u^{sep} is a natural transformation.

We claim that $(\_)^{\text{sep}}$ maps identities to identities, i.e. for all presheaves F in $\text{PSH}((\mathcal{C}, \mathcal{T}), \mathcal{D})$, the identity transformation 1_F will be mapped to $1_{F^{\text{sep}}}$, due to the following: For all objects U in $\mathcal{C}$ and all covering families $\mathfrak{U}$ of U, we have that $1_F U_{\mathfrak{U}\mathcal{H}} = 1_{\mathcal{H}_U(\mathfrak{U}, F)}$ is the identity morphism, therefore as $(1_F)^{\text{sep}}U = \text{colim}_{\text{Cov}^0(U)^{\text{op}}} 1_{\mathcal{H}_U(\_, F)}$, we have that $(1_F)^{\text{sep}}U = 1_{F^{\text{sep}}U}$.

Let $u : F \to G$ and $v : G \to H$ be natural transformations in $\text{PSH}((\mathcal{C}, \mathcal{T}), \mathcal{D})$. Then for all objects U in $\mathcal{C}$ and covering families $\mathfrak{U}$ of U, we have that $(v \circ u)U_{\mathfrak{U}\mathcal{H}} = vU_{\mathfrak{U}\mathcal{H}} \circ uU_{\mathfrak{U}\mathcal{H}} : \mathcal{H}_U(\mathfrak{U}, F) \to \mathcal{H}_U(\mathfrak{U}, H)$ and therefore $(v \circ u)^{\text{sep}}U = v^{\text{sep}}U \circ u^{\text{sep}}U$ due to the functoriality of colimits. We therefore have $(v \circ u)^{\text{sep}} = v^{\text{sep}} \circ u^{\text{sep}}$.

Altogether we have defined a functor $(\_)^{\text{sep}} : \text{PSH}((\mathcal{C}, \mathcal{T}), \mathcal{D}) \to \text{PSH}((\mathcal{C}, \mathcal{T}), \mathcal{D})$, which we call a *separation functor*.

4.2.13 Definition (Separation Morphisms) (III.2.2.9 [Mor20]**):** Let F be a presheaf in $\text{PSH}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ and let U be an object in $\mathcal{C}$. We know that $F^{\text{sep}}U = \text{colim}_{\text{Cov}^0(U)^{\text{op}}} \mathcal{H}_U(\_, F)$, with the accompanying colimit morphisms $(\phi_{U,\mathfrak{U}} : \mathcal{H}_U(\mathfrak{U}, F) \to F^{\text{sep}}U)_{\mathfrak{U} \in \text{Ob}(\text{Cov}^0(U))}$, forms a filtered colimit due to (**4.2.10**(b)). We then define the morphisms $\psi(F)_U : FU \to F^{\text{sep}}U$ in $\mathcal{D}$ as $\phi_{U,\mathfrak{U}} \circ \iota_{U,\mathfrak{U}}$ for any covering family $\mathfrak{U}$ of U, with $\iota_{U,\mathfrak{U}} : FU \to \mathcal{H}_U(\mathfrak{U}, F)$ from (**4.2.3**). Note that due to the properties of the colimit cocone $F^{\text{sep}}U$ and $\phi_{U,\mathfrak{U}}$, we have that this definition of $\psi(F)_U$ does not depend on the choice of the covering family $\mathfrak{U}$ of U.

For natural transformation $\psi(F) : F \to F^{\text{sep}}$: We claim that the morphisms $(\psi(F)_U)_{U \in \text{Ob}(\mathcal{C})}$ create a natural transformation $\psi(F) : F \to F^{\text{sep}}$ in $\text{PSH}((\mathcal{C}, \mathcal{T}), \mathcal{D})$. For a morphism $f : U \to V$ in $\mathcal{C}$, a covering family $\mathfrak{V}$ of V and the covering family $f^*\mathfrak{V}$ of U, we have:

$$\begin{aligned} \psi(F)_U \circ Ff &= \phi_{U,f^*\mathfrak{U}} \circ \iota_{U,f^*\mathfrak{U}} \circ Ff, \\ &= \phi_{U,f^*\mathfrak{U}} \circ f_{\mathfrak{V}\mathcal{H}} \circ \iota_{V,\mathfrak{V}}, \\ &= F^{\text{sep}}f \circ \phi_{V,\mathfrak{V}} \circ \iota_{V,\mathfrak{V}} = F^{\text{sep}}f \circ \psi(F)_V, \end{aligned}$$

with $\iota_{U,\mathfrak{U}}$ from (**4.2.3**) and $f_{\mathfrak{V}\mathcal{H}}$ from (**4.2.12**).

For natural transformation $\psi : 1_{\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})} \to (\_)^{\mathrm{sep}}$: Our definition of $\psi(F)$ for all presheaves F can be extended further into a natural transformation $\psi : 1_{\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})} \to (\_)^{\mathrm{sep}}$ of functors in $\mathrm{Func}(\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D}), \mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D}))$. We define ψ by the collection of natural transformations $(\psi(F) : F \to F^{\mathrm{sep}})_{F\in \mathrm{Ob}(\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D}))}$. To show that this is a well-defined transformation, we must show that for all natural transformations $u : F \to G$ in $\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})$, and for all objects U in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} FU & \xrightarrow{\psi(F)_U} & F^{\mathrm{sep}}U \\ {\scriptstyle uU}\downarrow & & \downarrow{\scriptstyle u^{\mathrm{sep}}U} \\ GU & \xrightarrow{\psi(G)_U} & G^{\mathrm{sep}}U \end{array} .$$

For any covering family $\mathfrak{U}$ of U, we have that $\psi(F)_U = \phi^F_{U,\mathfrak{U}} \circ \iota^F_{U,\mathfrak{U}}$ and $\psi(G)_U = \phi^G_{U,\mathfrak{U}} \circ \iota^G_{U,\mathfrak{U}}$ (with the defined morphisms $\phi_{U,\mathfrak{U}}$ and $\iota_{U,\mathfrak{U}}$, explicitly adding F and G to avoid confusion). Then we have:

$$u^{\mathrm{sep}}U \circ \psi(F)_U = u^{\mathrm{sep}}U \circ \phi^F_{U,\mathfrak{U}} \circ \iota^F_{U,\mathfrak{U}},$$

Due to the functoriality of colimits and the construction of $(\_)^{\mathrm{sep}}$, we have:

$$= \phi^G_{U,\mathfrak{U}} \circ uU_{\mathfrak{U}\mathcal{H}} \circ \iota^F_{U,\mathfrak{U}},$$

Due to the construction of $uU_{\mathfrak{U}\mathcal{H}}$, we have:

$$= \phi^G_{U,\mathfrak{U}} \circ \iota^G_{U,\mathfrak{U}} \circ uU = \psi(G)_U \circ uU.$$

4.2.14 Theorem (Sheafifications) (00WB [JC21]**), (III.2.2.10** [Mor20]**):** Let F be a presheaf in $\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})$, then the following statements are true:

(a) F^{sep} is a separated presheaf.

(b) If F is a separated presheaf, then $\psi(F) : F \to F^{\mathrm{sep}}$ is a monomorphism of presheaves.

(c) If F is a seperated presheaf, then F^{sep} is a sheaf in $\mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$.

(d) $F^{\mathrm{sh}} = (F^{\mathrm{sep}})^{\mathrm{sep}}$ is a sheaf in $\mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$.

(e) If F is a sheaf, then $\psi(F) : F \to F^{\mathrm{sep}}$ is an isomorphism of presheaves.

(f) The composition of functors $(\_)^{\mathrm{sh}} = (\_)^{\mathrm{sep}} \circ (\_)^{\mathrm{sep}} : \mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D}) \to \mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$ defines a *sheafification functor* and $\psi'(F) = \psi(F^{\mathrm{sep}}) \circ \psi(F) : F \to F^{\mathrm{sh}}$ defines a natural transformation from F to its *sheafification* F^{sh}. $(\_)^{\mathrm{sh}}$ also induces a natural transformation $\psi' : 1_{\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})} \to (\_)^{\mathrm{sh}}$.

Proof: For (a): Let U be an object of $\mathcal{C}$ and $\mathfrak{U} = (u_i : U_i \to U)_{i\in I}$ be a covering family of U, then we claim that the separation map $\iota^{\mathrm{sep}}_{U,\mathfrak{U}} : F^{\mathrm{sep}}U \to \mathcal{H}_U(\mathfrak{U}, F^{\mathrm{sep}})$, induced by $s \mapsto (F^{\mathrm{sep}}u_i(s))_{i\in I}$, is a monomorphism, which is enough due to (**4.2.3**). To show this, let $s, t \in F^{\mathrm{sep}}U$ such that $(F^{\mathrm{sep}}u_i(s))_{i\in I} = (F^{\mathrm{sep}}u_i(t))_{i\in I}$, i.e. for all $i \in I$ we have $F^{\mathrm{sep}}u_i(s) = F^{\mathrm{sep}}u_i(t) \in F^{\mathrm{sep}}U_i$. Due to the properties of filtration in $\mathrm{Set}_{\mathcal{U}}$, as explained in (**1.5.16**), we can identify s and t as elements of the underlying sets:

$$\mathrm{For}(F^{\mathrm{sep}}U) = \left(\coprod_{\mathfrak{V}\in \mathrm{Ob}(\mathrm{Cov}^0(U)^{\mathrm{op}})} \mathrm{For}(\mathcal{H}_U(\mathfrak{V}, F)) \right) / \sim' .$$

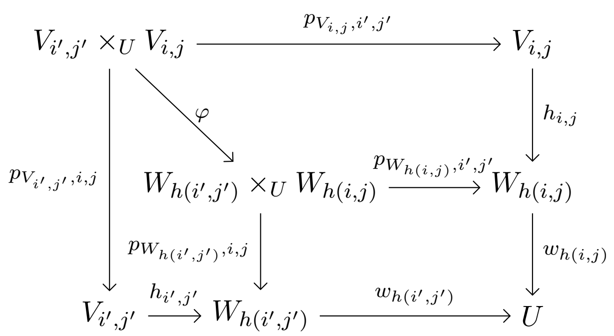
Therefore, there exists a covering family $\mathfrak{W} = (w_k : W_k \to U)_{k\in K}$ of U, with the refinement $f : \mathfrak{W} \to \mathfrak{U}$, such that s and t are represented by $(s_k)_{k\in K}$ and $(t_k)_{k\in K}$ in $\mathcal{H}_U(\mathfrak{W}, F)$. Similarly for all $i \in I$, we can identify $F^{\mathrm{sep}}u_i(s)$, $F^{\mathrm{sep}}u_i(t)$ as elements of the underlying sets:

$$\mathrm{For}(F^{\mathrm{sep}}U_i) = \left(\coprod_{\mathfrak{U}_i \in \mathrm{Ob}(\mathrm{Cov}^0(U_i)^{\mathrm{op}})} \mathrm{For}(\mathcal{H}_{U_i}(\mathfrak{U}_i, F)) \right) / \sim' .$$

Therefore for all $i \in I$, there exists a covering family $\mathfrak{V}_i = (v_{i,j} : V_{i,j} \to U_i)_{j \in J_i}$, with the refinement $g_i : \mathfrak{V}_i \to u_i^* \mathfrak{W}$, such that the equivalence classes $F^{\mathrm{sep}}u_i(s)$ and $F^{\mathrm{sep}}u_i(t)$ are both represented by the same element $(w_{i,j})_{j \in J_i} \in \mathcal{H}_{U_i}(\mathfrak{V}_i, F)$.

Let $L = \prod_{i \in I} J_i$ be an index set and let $\mathfrak{X} = (u_i \circ v_{i,j} : V_{i,j} \to U)_{(i,j) \in L}$ be a covering family of U, which exists due to (**4.2.1**(ii)). It can be easily checked that the morphisms of covering families $(g_i : \mathfrak{V}_i \to u_i^* \mathfrak{W})_{i \in I}$ composed with $(u_{i\mathfrak{W}} : u_i^* \mathfrak{W} \to \mathfrak{W})_{i \in I}$, as defined in (**4.2.12**), induce a refinement $h : \mathfrak{X} \to \mathfrak{W}$ with the morphisms $(h_{i,j} = p_{W_{g_i(j)}} \circ (g_i)_j)_{i,j \in L}$ with the projection $p_{W_{g_i(j)}} : W_{g_i(j)} \times_U U_i \to W_{g_i(j)}$. It is clear that $F^{\mathrm{sep}}u_i(s)$ is represented by $(Fp_{W_{g_i(j)}}(s_{g_i(j)}))_{j \in J_i}$ on $u_i^* \mathfrak{W}$ and that we have $(F(g_i)_j \circ Fp_{W_{g_i(j)}}(s_{g_i(j)}))_{j \in J} = Fh_{i,j}(s_{h(i,j)})$ on $\mathfrak{X}$. This implies $(w_{i,j})_{(i,j)} = Fh_{i,j}(s_{h(i,j)})$ for all $(i,j) \in L$.

We then claim that $(w_{i,j})_{(i,j) \in L} \in \mathcal{H}_U(\mathfrak{X}, F)$: For any two indices $(i,j), (i',j') \in L$, we define all the corresponding fiber products and projections:



,

with φ being the induced morphism from the fiber product. When we apply F to the above diagram and invert the arrows, we get:

$$\begin{aligned} Fp_{V_{i',j'},i,j}(w_{i',j'}) &= Fp_{V_{i',j'},i,j} \circ Fh_{i',j'}(s_{h(i',j')}), \\ &= F\varphi \circ Fp_{W_{h(i',j')},i,j}(s_{h(i',j')}), \end{aligned}$$

Since $(s_k)_{k \in K} \in \mathcal{H}_U(\mathfrak{W}, F)$, we then have that:

$$\begin{aligned} &= F\varphi \circ Fp_{W_{h(i,j)},i',j'}(s_{h(i,j)}), \\ &= Fp_{V_{i,j},i',j'} \circ Fh_{i,j}(s_{h(i,j)}) = Fp_{V_{i,j},i',j'}(w_{i,j}). \end{aligned}$$

Therefore we have $(w_{i,j})_{(i,j) \in L} \in \mathcal{H}_U(\mathfrak{X}, F)$ as claimed.

We can use (**1.5.16**) again to see that s and t, represented by equivalence classes in $\mathrm{For}(F^{\mathrm{sep}}U)$, are both represented by the same representative $(w_{i,j})_{(i,j) \in L} \in \mathcal{H}_U(\mathfrak{X}, F)$, which implies $s = t$. This implies that $\iota^{\mathrm{sep}}_{U,\mathfrak{U}}$ is a monomorphism as claimed.

For (b): Due to (**1.3.4**(a)), it is enough to show that $\psi(F)_U : FU \to F^{\mathrm{sep}}U$ is a monomorphism for any object U in $\mathcal{C}$. Let $s, t \in FU$, such that $\psi(F)_U(s) = \psi(F)_U(t) \in F^{\mathrm{sep}}U$, then due to the filtration in (**1.5.16**), there exists a covering family $\mathfrak{V} = (v_j : V_j \to U)_{j \in J}$ of U, such that $\psi(F)_U(s)$ and $\psi(F)_U(t)$ have the same representative $(w_j)_{j \in J} \in \mathcal{H}_U(\mathfrak{V}, F)$. Due to F being separated, the separation map $\iota_{U,\mathfrak{V}} : FU \to \mathcal{H}_U(\mathfrak{V}, F)$ is a monomorphism due to (**4.2.3**) and since $\iota_{U,\mathfrak{V}}(s) = \iota_{U,\mathfrak{V}}(t) = (w_j)_{j \in J}$, we have $s = t$.

For (c): Let U be any object in $\mathcal{C}$ and $\mathfrak{U} = (u_i : U_i \to U)_{i \in I}$ be any covering family of U, then we claim that the separation map $\iota^{\mathrm{sep}}_{U,\mathfrak{U}} : F^{\mathrm{sep}}U \to \mathcal{H}_U(\mathfrak{U}, F^{\mathrm{sep}})$ is an isomorphism, which is enough due to (**4.2.3**). Due to (a) we already have that $\iota^{\text{sep}}_{U,\mathfrak{U}}$ is a monomorphism. It is therefore enough to show that $\text{For}(\iota^{\text{sep}}_{U,\mathfrak{U}})$ is an epimorphism, as a function that is a monomorphism and an epimorphism in $\text{S}\textsc{et}_{\mathcal{U}}$ (i.e. injective and surjective) is also an isomorphism (i.e. bijective), implying that $\iota^{\text{sep}}_{U,\mathfrak{U}}$ is an isomorphism. Let $s = (s_i)_{i\in I} \in \mathcal{H}_U(\mathfrak{U}, F^{\text{sep}})$ with $s_i \in F^{\text{sep}}U_i$ for all $i \in I$. For every $i \in I$, s_i is represented due to (**1.5.16**) by an element $(w_{i,j})_{j\in J_i} \in \mathcal{H}_{U_i}(\mathfrak{V}_i, F)$, whereby $\mathfrak{V}_i = (v_{i,j} : V_{i,j} \to U_i)_{j\in J_i}$ is a covering family of U_i. Like in (a) we define $L = \prod_{i\in I} J_i$, the covering family $\mathfrak{X} = (u_i \circ v_{i,j} : V_{i,j} \to U)_{(i,j)\in L}$ and combine the elements $(w_{i,j})_{j\in J_i}$ into $(w_{i,j})_{(i,j)\in L} \in \mathcal{H}_U(\mathfrak{X}, F)$. Due to (**1.5.16**), we have that $(w_{i,j})_{i\in I, j\in J_i}$ is a representation of an element $w \in F^{\text{sep}}U$.

We have then $\iota^{\text{sep}}_{U,\mathfrak{U}}(w) = (w_i)_{i\in I} \in \mathcal{H}_U(\mathfrak{U}, F^{\text{sep}})$ with $w_i \in F^{\text{sep}}U_i$. Per construction we have that w_i is represented by $(w_{i,j})_{j\in J_i} \in \mathcal{H}_{U_i}(\mathfrak{V}_i, F)$ which also represents s_i, i.e. $w_i = s_i$ and thus $\iota^{\text{sep}}_{U,\mathfrak{U}}(w) = s$ implies that $\iota^{\text{sep}}_{U,\mathfrak{U}}$ is an epimorphism and that F^{sep} is a sheaf.

For (d): Due to (a), F^{sep} is a separated presheaf, due to (c) $F^{\text{sh}} = (F^{\text{sep}})^{\text{sep}}$ is a sheaf in $\text{S}\textsc{h}((\mathcal{C}, \mathcal{T}), \mathcal{D})$.

For (e): Due to (**1.3.4**(a)), it is enough to show that $\psi(F)_U : FU \to F^{\text{sep}}U$ is an isomorphism for all objects U in $\mathcal{C}$. Due to (b), $\psi(F)_U$ is already a monomorphism and due to the same argument involving $\text{S}\textsc{et}_{\mathcal{U}}$ as seen in (c), it is enough to show that $\text{For}(\psi(F)_U)$ is an epimorphism. Let $s \in F^{\text{sep}}U$ be an element represented by $(s_i)_{i\in I} \in \mathcal{H}_U(\mathfrak{U}, F)$, where $\mathfrak{U} = (u_i : U_i \to U)_{i\in I}$ is a covering family of U and $s_i \in FU_i$ for all $i \in I$. Since F is a sheaf, the mapping $\iota_{U,\mathfrak{U}} : FU \to \mathcal{H}_U(\mathfrak{U}, F)$ is an isomorphism due to (**4.2.3**) and thus there exists $w \in FU$ such that $\iota_{U,\mathfrak{U}}(w) = (s_i)_{i\in I}$. In (**4.2.13**) we defined the colimit morphism $\phi_{U,\mathfrak{U}} : \mathcal{H}_U(\mathfrak{U}, F) \to F^{\text{sep}}U$ and it is clear that $\phi_{U,\mathfrak{U}}((s_i)_{i\in I}) = s$, i.e. $\psi_U(w) = \phi_{U,\mathfrak{U}} \circ \iota_{U,\mathfrak{U}}(w) = s$. Thus $\psi(F)_U$ is an epimorphism.

For (f): Due to (**4.2.12**) and (d), we know that $(\_)^{\text{sh}} : \text{PS}\textsc{h}((\mathcal{C}, \mathcal{T}), \mathcal{D}) \to \text{S}\textsc{h}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ is a well-defined functor. Furthermore due to (**4.2.13**), we know that $\psi(F) : F \to F^{\text{sep}}$ is a natural transformation. Therefore, $\psi'(F) = \psi(F^{\text{sep}}) \circ \psi(F) : F \to F^{\text{sh}}$ is also a natural transformation.

Due to the construction of the natural transformation $\psi : 1_{\text{PS}\textsc{h}((\mathcal{C},\mathcal{T}),\mathcal{D})} \to (\_)^{\text{sep}}$ in (**4.2.13**), it is clear that $\psi'(\_) = \psi((\_)^{\text{sep}}) \circ \psi(\_) : 1_{\text{PS}\textsc{h}((\mathcal{C},\mathcal{T}),\mathcal{D})} \to (\_)^{\text{sh}}$ is also a natural transformation. □

4.2.15 Corollary (Sheafifications) (III.2.2.10 [Mor20]**):** Let G be a sheaf in $\text{S}\textsc{h}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ and let $u : F \to G$ be a morphism of presheaves, then there exists unique natural transformation $u' : F^{\text{sh}} \to G$ such that $u = u' \circ \psi'(F)$, whereby $\psi'(F) = \psi(F^{\text{sep}}) \circ \psi(F)$.

Proof: For construction of u': Due to (**4.2.13**) and (**4.1.14**(e)), we know that the following diagrams commute:

$$\begin{array}{ccc} F & \xrightarrow{\psi(F)} & F^{\text{sep}} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u^{\text{sep}}} \\ G & \xrightarrow[\psi(G)]{} & G^{\text{sep}} \end{array} \quad , \qquad \begin{array}{ccc} F & \xrightarrow{\psi'(F)} & F^{\text{sh}} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u^{\text{sh}}} \\ G & \xrightarrow[\psi'(G)]{} & G^{\text{sh}} \end{array} \quad .$$

Thus, using (**4.1.14**(e)) we have that $\psi'(G)$ is an isomorphism and thus $u' = (\psi'(G))^{-1} \circ u^{\text{sh}}$ gives us the desired natural transformation.

For u' being unique: Let $u'' : F^{\text{sh}} \to G$ be a morphism of presheaves such that $u = u'' \circ \psi'(F)$. In order to show that $u' = u''$, we must show that for all objects U in $\mathcal{C}$, we have that for the morphisms $u'U, u''U : F^{\text{sh}}U \to GU$ and an element $s \in F^{\text{sh}}U$, we have $u'U(s) = u''U(s)$. Since $F^{\text{sh}}U = (F^{\text{sep}})^{\text{sep}}U$ is a filtered colimit of a filtered colimit, we apply (**1.5.16**) twice to the underlying set of $F^{\text{sh}}U$. There exists a covering family $\mathfrak{U} = (u_i : U_i \to U)_{i\in I}$ of U such that there exists an element $(s_i)_{i\in I} \in \mathcal{H}_U(\mathfrak{U}, F^{\text{sep}})$ that represents s, then for each $i \in I$, $s_i \in F^{\text{sep}}U_i$ can be further represented by a $(s_{i,j})_{j\in J_i} \in \mathcal{H}_U(\mathfrak{V}_i, F)$ for a covering morphism $\mathfrak{V}_i = (v_{i,j} : V_{i,j} \to U_i)_{j\in J_i}$ of U_i. Let $L = \prod_{i\in I} J_i$ and let $\mathfrak{X} = (u_i \circ v_{i,j} : V_{i,j} \to U)_{(i,j)\in L}$ be a covering family of U, we then have $(s_{i,j})_{(i,j)\in L} \in \mathcal{H}_U(\mathfrak{X}, F)$ just as in the proof of (**4.1.14**(a)).

Let (i, j) be an index in L, due to the filtered colimit structure of $F^{\text{sh}}V_{i,j}$ given by (**1.5.16**), and due to $F^{\text{sh}}(u_i \circ v_{i,j})$ being constructed as a colimit morphism, it is clear that $F^{\text{sh}}(u_i \circ v_{i,j})(s) = \psi'(F)_{V_{i,j}}(s_{i,j})$. Therefore for all $(i, j) \in K$, we have that:

$$\begin{aligned} G(u_i \circ v_{i,j})u'U(s) = u'V_{i,j}(F^{\text{sh}}(u_i \circ v_{i,j})(s)) &= u'V_{i,j}\psi'(F)_{V_{i,j}}(s_{i,j}), \\ &= u''V_{i,j}\psi'(F)_{V_{i,j}}(s_{i,j}) = G(u_i \circ v_{i,j})u''U(s). \end{aligned}$$

Due to G being a sheaf, which implies that $GU \cong \mathcal{H}_U(\mathfrak{X}, G)$, it is clear that the above equalities imply that $u'U(s) = u''U(s)$ and thus $u' = u''$ as claimed. □

4.2.16 Corollary (Inclusions and Sheafifications are Adjoint) (III.2.2.11 [Mor20]**):** We have that the inclusion functor $\iota : \textsc{Sh}((\mathcal{C}, \mathcal{T}), \mathcal{D}) \to \textsc{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ and the sheafification functor $(\_)^{\text{sh}} : \textsc{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D}) \to \textsc{Sh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ form an adjoint pair $((\_)^{\text{sh}}, \iota)$.

Proof: Let F be a $\mathcal{D}$-valued presheaf on $(\mathcal{C}, T)$ and let G be a $\mathcal{D}$-valued sheaf on $(\mathcal{C}, T)$. We define the mapping $v_{F,G} : \text{Hom}_{\textsc{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})}(F, \iota(G)) \to \text{Hom}_{\textsc{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})}(F^{\text{sh}}, G)$, $(u : F \to G) \mapsto (u' : F^{\text{sh}} \to G)$ as the induced natural transformation seen in (**4.2.15**). Furthermore, due to the uniqueness of u' as proven in (**4.2.15**), we have that the mapping $t_{F,G} : \text{Hom}_{\textsc{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})}(F^{\text{sh}}, G) \to \text{Hom}_{\textsc{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})}(F, \iota(G))$, $u' \mapsto u' \circ \psi'(F)$ is an inverse to $v_{F,G}$ and thus $v_{F,G}$ is an isomorphism in $\textsc{Set}_{\mathcal{U}}$ as it is bijective.

In order to check that there exists a natural isomorphism between $\text{Hom}_{\textsc{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})}(\_, \iota(\_))$ and $\text{Hom}_{\textsc{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})}((\_)^{\text{sh}}, \_)$, let $a : E \to F$ be a morphism of presheaves and $b : G \to H$ be a morphism of sheaves, then for any morphism of presheaves $u : F \to G$, we have that $b \circ u \circ a : E \to H$ maps to $v_{E,H}(b \circ u \circ a) : E^{\text{sh}} \to H$, whereby $b \circ u \circ a = v_{E,H}(b \circ u \circ a) \circ \psi'(E)$. However, for $v_{F,G}(u) : F^{\text{sh}} \to G$ which fulfills $u = v_{F,G}(u) \circ \psi'(F)$, we have that $b \circ u \circ a = b \circ v_{F,G}(u) \circ \psi'(F) \circ a = (b \circ v_{F,G}(u) \circ a^{\text{sh}}) \circ \psi'(E)$ due to ψ' being a natural transformation as seen in (**4.1.14**(e)). Due to the uniqueness of $v_{E,H}(b \circ u \circ a)$, we have $v_{E,H}(b \circ u \circ a) = b \circ v_{F,G}(u) \circ a^{\text{sh}}$. This natural isomorphism gives us the adjunction $((\_)^{\text{sh}}, \iota)$ as claimed. □

4.2.17 Corollary (Limits and Colimits of Sheaves) (III.2.2.13 [Mor20]**):** The following statements apply:

(a) The inclusion functors $\textsc{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D}) \to \textsc{PSh}((\mathcal{C}, \mathcal{T}))$ and $\textsc{Sh}((\mathcal{C}, \mathcal{T}), \mathcal{D}) \to \textsc{Sh}((\mathcal{C}, \mathcal{T}))$, induced by the forgetful functor $\text{For} : \mathcal{D} \to \textsc{Set}_{\mathcal{U}}$, commute with $\mathcal{U}$-small limits and $\mathcal{U}$-small filtered colimits.

Now let $\mathcal{D}$ be a good concrete $\mathcal{U}$-category that has all $\mathcal{U}$-small limits and colimits. Then the following applies:

(b) $\textsc{Sh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ has all $\mathcal{U}$-small limits and colimits.

(c) The inclusion functor $\iota : \textsc{Sh}((\mathcal{C}, \mathcal{T}), \mathcal{D}) \to \textsc{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ commutes with all $\mathcal{U}$-small limits.

(d) The sheafification functor $(\_)^{\text{sh}} : \textsc{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D}) \to \textsc{Sh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ commutes with all $\mathcal{U}$-small colimits as well as finite limits.

Proof: For (a): By definition of $\mathcal{D}$ being a good concrete $\mathcal{U}$-category, $\mathcal{D}$ has an associated forgetful functor $\text{For} : \mathcal{D} \to \textsc{Set}_{\mathcal{U}}$ that commutes with $\mathcal{U}$-small limits and $\mathcal{U}$-small filtered colimits. Since limits and colimits in $\textsc{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ and $\textsc{PSh}((\mathcal{C}, \mathcal{T}))$ are determined object-wise due to (**I.5.3.1** [Mor20]), we can determine if $\textsc{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D}) \to \textsc{PSh}((\mathcal{C}, \mathcal{T}))$ commutes with $\mathcal{U}$-small limits and $\mathcal{U}$-small filtered colimits if For does this, which is the case. We apply same argument above but with sheaves, which implies that the inclusion $\textsc{Sh}((\mathcal{C}, \mathcal{T}), \mathcal{D}) \to \textsc{Sh}((\mathcal{C}, \mathcal{T}))$ also commutes with $\mathcal{U}$-small limits and $\mathcal{U}$-small filtered colimits.

For $\textsc{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ having $\mathcal{U}$-small limits and colimits: We already know that due to $\mathcal{D}$ having all $\mathcal{U}$-small limits and colimits and due to limits and colimits in $\textsc{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ being determined object-wise, we have that $\textsc{PSh}((\mathcal{C}, \mathcal{T}), \mathcal{D})$ has all $\mathcal{U}$-small limits and colimits.

For (b), $\mathcal{U}$-small limits: In (**4.2.12**($\star\star$)), we saw that for a morphism between presheaves $u : F \to G$ in $\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})$ and for an object U in $\mathcal{C}$ with a covering family $\mathfrak{U}$, we can induce a morphism $uU_{\mathfrak{U}\mathcal{H}} : \mathcal{H}_U(\mathfrak{U}, F) \to \mathcal{H}_U(\mathfrak{U}, G)$. It is easy to check that these morphisms are functorial in u, i.e. that for another morphism of presheaves $v : G \to H$ that $vU_{\mathfrak{U}\mathcal{H}} \circ uU_{\mathfrak{U}\mathcal{H}} = (v \circ u)U_{\mathfrak{U}\mathcal{H}}$. Therefore, we can induce a functor $\mathcal{H}_U(\mathfrak{U}, \_) : \mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D}) \to \mathcal{D}$.

Let $D : \mathcal{I} \to \mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$ be a $\mathcal{U}$-small diagram, then we know that a limit $F = \lim_{\mathcal{I}} D$ with its limit cone exists in $\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})$. It is enough to show that F is a sheaf in $\mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$ since $\mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$ is a full subcategory of $\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})$. For any object U in $\mathcal{C}$ with a covering family $\mathfrak{U}$, we have the functor $\mathcal{H}_U(\mathfrak{U}, \_)$ which maps presheaves to objects in $\mathcal{D}$, that are defined through equalizers, which are $\mathcal{U}$-small limits. Since $\mathcal{U}$-small limits commute with each other, as seen in (**I.5.4.1** [Mor20]), we have that $\mathcal{H}_U(\mathfrak{U}, \_)$ maps $F = \lim_{\mathcal{I}} D$ to $\mathcal{H}_U(\mathfrak{U}, F) = \lim_{\mathcal{I}} \mathcal{H}_U(\mathfrak{U}, D(\_))$.

As we saw in (**4.2.3**), it is enough to show that for every object U in $\mathcal{C}$ and for every covering family $\mathfrak{U}$ of U, we have that $\iota_{U,\mathfrak{U}} : FU \to \mathcal{H}_U(\mathfrak{U}, F)$ is an isomorphism. Since for every object i in $\mathcal{I}$, we have that $D(i)$ is a sheaf, we have that the separation morphisms of $\iota^i_{U,\mathfrak{U}} : D(i)U \to \mathcal{H}_U(\mathfrak{U}, D(i))$ are isomorphisms. These morphisms induce a natural isomorphism $u : D(\_)U \to \mathcal{H}_U(\mathfrak{U}, D(\_))$ between functors, which implies that the morphism $\lim_{\mathcal{I}} u = \iota_{U,\mathfrak{U}} : FU \to \mathcal{H}_U(\mathfrak{U}, F)$ is an isomorphism, due to $\iota_{U,\mathfrak{U}}$ having an inverse morphism $\lim_{i \in \mathrm{Ob}(\mathcal{I})}(u^{-1})$. Therefore F is a sheaf, making F the limit of the diagram in $\mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$ as well as $\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})$. $\mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$ therefore has all $\mathcal{U}$-small limits.

For (c): The above argument also directly shows that the inclusion functor ι commutes with all $\mathcal{U}$-small limits.

For (b), $\mathcal{U}$-small colimits: For any $\mathcal{U}$-small diagram $D : \mathcal{I} \to \mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$, we know that a colimit $F = \mathrm{colim}_{\mathcal{I}} D$ exists in $\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})$. We claim that the sheafification F^{sh} is the limit of the same diagram but in $\mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$. We have the following natural transformations due to the adjunction $((\_)^{\mathrm{sh}}, \iota)$ from (**4.2.16**):

$$\mathrm{Hom}_{\mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})}(F^{\mathrm{sh}}, \_) \cong \mathrm{Hom}_{\mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})}(F, \iota(\_)),$$

By generalizing the bijection from (**1.5.10**) into a natural isomorphism and by using the fact that F is a colimit, we have that:

$$\cong \lim_{i \in \mathrm{Ob}(\mathcal{I}^{\mathrm{op}})} \mathrm{Hom}_{\mathrm{PSh}((\mathcal{C},\mathcal{T}))}(D^{\mathrm{op}}(i), \iota(\_)),$$

Due to sheaves forming a full subcategory of presheaves and due to the adjoint pair $((\_)^{\mathrm{sh}}, \iota)$ from (**4.2.16**), we have:

$$\cong \lim_{i \in \mathrm{Ob}(\mathcal{I}^{\mathrm{op}})} \mathrm{Hom}_{\mathrm{Sh}((\mathcal{C},\mathcal{T}))}(D^{\mathrm{op}}(i)^{\mathrm{sh}}, \_),$$
$$\cong \mathrm{Hom}_{\mathrm{Sh}((\mathcal{C},\mathcal{T}))}(\mathrm{colim}_{\mathcal{I}}((\_)^{\mathrm{sh}} \circ D), \_).$$

Therefore with the help of the *dual Yoneda lemma* in (**1.4.9**), we can imply that F^{sh} is the colimit in $\mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$ of the diagram $D \cong (\_)^{\mathrm{sh}} \circ D$.

For (d), $\mathcal{U}$-small colimits: For any diagram $D : \mathcal{I} \to \mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})$, we can still apply the same arguments made in (b) to find that $F^{\mathrm{sh}} = (\mathrm{colim}_{\mathcal{I}} D)^{\mathrm{sh}}$ is the colimit of $(\_)^{\mathrm{sh}} \circ D$ in $\mathrm{Sh}((\mathcal{C},\mathcal{T}),\mathcal{D})$. Therefore $(\_)^{\mathrm{sh}}$ commutes with $\mathcal{U}$-small colimits.

For (d), finite limits: We claim that $(\_)^{\mathrm{sep}}$ commutes with finite limits, which would imply that $(\_)^{\mathrm{sh}} = (\_)^{\mathrm{sep}} \circ (\_)^{\mathrm{sep}}$ commutes with finite limits. Let U be an object in $\mathcal{C}$ and $D : \mathcal{I} \to \mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D})$ be a finite diagram. Since limits of presheaves exist and are determined objectwise, as seen in (**I.5.3.1** [Mor20]), it is enough to show that $(\lim_{\mathcal{I}} D)^{\mathrm{sep}} U \cong \lim_{i \in \mathrm{Ob}(\mathcal{I})}(D^{\mathrm{sep}}(i)U)$. We first claim that $\mathcal{H}_U(\mathfrak{U}, \_) : \mathrm{PSh}((\mathcal{C},\mathcal{T}),\mathcal{D}) \to \mathcal{D}$ commutes with finite limits for all covering families $\mathfrak{U}$ of U: This is clear as $\mathcal{H}_U(\mathfrak{U}, F)$ is constructed as a limit (specifically an equalizer) for all presheaves F, and due to the fact that limits commute with each other, as seen in (**I.5.4.1** [Mor20]). Therefore we have:

$$\lim_{i\in\mathrm{Ob}(\mathcal{I})}(D^{\mathrm{sep}}(i)U) = \lim_{i\in\mathrm{Ob}(\mathcal{I})}(\mathrm{colim}_{\mathrm{Cov}^0(U)^{\mathrm{op}}}\mathcal{H}_U(\_, D(i))),$$

Due to finite limits and $\mathcal{U}$-small filtered colimits being able to commute, as seen in (**I.5.6.4** [Mor20]), we have that:

$$\begin{aligned}
&\cong \mathrm{colim}_{\mathfrak{U}\in\mathrm{Ob}(\mathrm{Cov}^0(U)^{\mathrm{op}})}(\lim_{\mathcal{I}}\mathcal{H}_U(\mathfrak{U}, D(_))),\\
&\cong \mathrm{colim}_{\mathfrak{U}\in\mathrm{Ob}(\mathrm{Cov}^0(U)^{\mathrm{op}})}(\mathcal{H}_U(\mathfrak{U}, \lim_{\mathcal{I}}D(_))),\\
&= (\lim_{\mathcal{I}}D)^{\mathrm{sep}}U,
\end{aligned}$$

and the claim follows. □

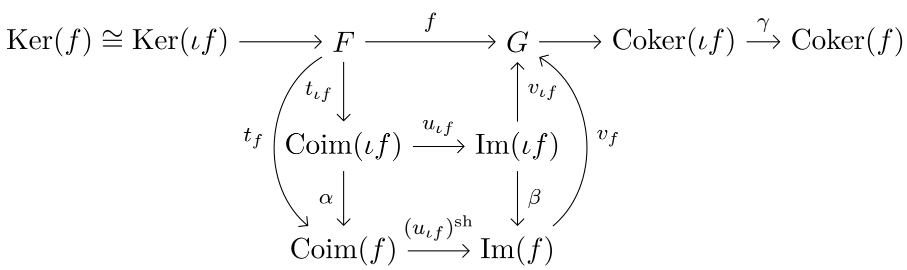
4.2.18 Corollary (Categories of Sheaves are Abelian Categories) (III.2.2.14 [Mor20]**):** Let $(\mathcal{C},\mathcal{T})$ be a $\mathcal{U}$-small Grothendieck pretopology with $\mathcal{U}$-small covering families and let R be a $\mathcal{U}$-ring, then the following applies:

(a) $\mathrm{S{\small H}}((\mathcal{C},\mathcal{T}),R)$ is an abelian $\mathcal{U}$-category.

(b) The inclusion functor $\iota : \mathrm{Sh}((\mathcal{C},\mathcal{T}),R) \to \mathrm{PSh}((\mathcal{C},\mathcal{T}),R)$ is left exact and the sheafification functor $(\_)^{\mathrm{sh}} : \mathrm{PSh}((\mathcal{C},\mathcal{T}),R) \to \mathrm{Sh}((\mathcal{C},\mathcal{T}),R)$ is exact.

Proof: For (a): As seen in (**2.3.2**(c)), we know that $\mathrm{PSh}((\mathcal{C},\mathcal{T}),R)$ is an abelian $\mathcal{U}$-category as ${}_R\mathrm{Mod}_{\mathcal{U}}$ is an abelian $\mathcal{U}$-category. Therefore, since $\mathrm{Sh}((\mathcal{C},\mathcal{T}),R)$ is a full subcategory of $\mathrm{PSh}((\mathcal{C},\mathcal{T}),R)$, we can define the addition of morphisms in $\mathrm{Sh}((\mathcal{C},\mathcal{T}),R)$ exactly like we do with $\mathrm{PSh}((\mathcal{C},\mathcal{T}),R)$, which clearly makes the composition $\circ$ bilinear. Furthermore, it is clear that finite biproducts and zero objects in $\mathrm{Sh}((\mathcal{C},\mathcal{T}),R)$ exist as $\mathrm{Sh}((\mathcal{C},\mathcal{T}),R)$ contains all $\mathcal{U}$-small limits and $\mathcal{U}$-small colimits due to (**4.2.17**(b)). This implies that $\mathrm{Sh}((\mathcal{C},\mathcal{T}),R)$ is an additive $\mathcal{U}$-category.

Since $\mathrm{Sh}((\mathcal{C},\mathcal{T}),R)$ contains all $\mathcal{U}$-small limits and $\mathcal{U}$-small colimits due to (**4.2.17**(b)), $\mathrm{Sh}((\mathcal{C},\mathcal{T}),R)$ contains all kernels and cokernels. Therefore for every morphism $f : F \to G$ in $\mathrm{Sh}((\mathcal{C},\mathcal{T}),R)$, we have that the canonical decomposition $f = v_f \circ u_f \circ t_f$ from (**2.2.5**) exists. It remains to be shown that u_f is an isomorphism: Since the inclusion functor $\iota : \mathrm{Sh}((\mathcal{C},\mathcal{T}),R) \to \mathrm{PSh}((\mathcal{C},\mathcal{T}),R)$ commutes with limits due to (**4.2.17**(c)), we have that $\mathrm{Ker}(f) \cong \mathrm{Ker}(\iota f)$ for $\mathrm{Ker}(f)$ being the kernel in $\mathrm{Sh}((\mathcal{C},\mathcal{T}),R)$ and $\mathrm{Ker}(\iota f)$ being the kernel in $\mathrm{PSh}((\mathcal{C},\mathcal{T}),R)$. Analogously, we have $\mathrm{Coker}(f) \cong (\mathrm{Coker}(\iota f))^{\mathrm{sh}}$ as sheafification commutes with colimits due to (**4.2.17**(d)). Similarly for the kernel morphism $\mathrm{Ker}(f) \to F$, we have $\mathrm{Coim}(f) \cong \mathrm{Coker}(\mathrm{Ker}(f) \to F) \cong (\mathrm{Coker}(\mathrm{Ker}(\iota f) \to F))^{\mathrm{sh}} \cong (\mathrm{Coim}(\iota f))^{\mathrm{sh}}$. For $\mathrm{Im}(f)$, we represent the cokernel morphism of f with $G \to \mathrm{Coker}(f)$, then we have $\mathrm{Im}(f) \cong \mathrm{Ker}(G \to \mathrm{Coker}(f)) \cong \mathrm{Ker}((G \to \mathrm{Coker}(\iota f))^{\mathrm{sh}})$. Due to sheafification commuting with finite limits, as seen in (**4.2.17**(d)), we have $\mathrm{Im}(f) \cong (\mathrm{Ker}(G \to \mathrm{Coker}(\iota f)))^{\mathrm{sh}} \cong (\mathrm{Im}(\iota f))^{\mathrm{sh}}$. As $\mathrm{PSh}((\mathcal{C},\mathcal{T}),R)$ is an abelian category, the canonical morphism $u_{\iota f} : \mathrm{Coim}(\iota f) \to \mathrm{Im}(\iota f)$, in the decomposition of $\iota f = v_{\iota f} \circ u_{\iota f} \circ t_{\iota f}$ as seen in (**2.2.5**), is an isomorphism. Therefore since (**4.2.17**(d)) implies that sheafification $(\_)^{\mathrm{sh}}$ commutes with kernels and cokernels, it can be shown that the following diagram commutes:



,

whereby the morphisms α, β and γ are the canonical morphisms from a presheaf to its sheafification, as seen in (**4.2.14**(f)), and the remaining morphisms are the canonical kernels and cokernels. The above diagram directly implies that $(u_{\iota f})^{\mathrm{sh}}$ fulfills the canonical decomposition $f = v_f \circ (u_{\iota f})^{\mathrm{sh}} \circ t_f$. This implies that $u_f = (u_{\iota f})^{\mathrm{sh}}$. Since $u_{\iota f}$ is an isomorphism, there exists an inverse morphism $v : \mathrm{Im}(\iota f) \to \mathrm{Coim}(\iota f)$ and since sheafification is a functor, it is clear that v^{sh} is an inverse to $(u_{\iota f})^{\mathrm{sh}}$ in $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$ and therefore $u_f = (u_{\iota f})^{\mathrm{sh}}$ is an isomorphism in $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$.

For (b): Due to how addition in $\mathrm{PSH}((\mathcal{C}, \mathcal{T}), R)$ and $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$ is defined, it is clear that the inclusion functor $\iota : \mathrm{PSH}((\mathcal{C}, \mathcal{T}), R) \to \mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$ is an additive functor. Similarly, as $(\_)^{\mathrm{sep}}$ is induced by colimits, it is also clearly an additive functor as colimit functors are additive due to (**2.1.8**(b)) and thus $(\_)^{\mathrm{sh}}$ is an additive functor.

Due to (**4.2.17**), we know that ι commutes with finite limits and that $(\_)^{\mathrm{sh}}$ commutes with finite limits as well as finite colimits. This implies that ι is left exact and that $(\_)^{\mathrm{sh}}$ is exact. ☐

4.2.19 Corollary (Categories of Sheaves are Grothendieck Abelian Categories) (III.2.2.15 [Mor20]**):** For a $\mathcal{U}$-small Grothendieck pretopology $(\mathcal{C}, \mathcal{T})$ with $\mathcal{U}$-small covering families, we have that $\mathrm{PSH}((\mathcal{C}, \mathcal{T}), R)$ and $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$ are Grothendieck abelian $\mathcal{U}$-categories, i.e. the following statements are true:

(a) $\mathrm{PSH}((\mathcal{C}, \mathcal{T}), R)$ and $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$ have generators.

(b) All $\mathcal{U}$-small colimits exist in $\mathrm{PSH}((\mathcal{C}, \mathcal{T}), R)$ and $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$.

(c) For a $\mathcal{U}$-small filtered category $\mathcal{I}$, the colimit functors $\mathrm{colim}_{\mathcal{I}}^{\mathrm{PSH}} : \mathrm{FUNC}(\mathcal{I}, \mathrm{PSH}((\mathcal{C}, \mathcal{T}), R)) \to \mathrm{PSH}((\mathcal{C}, \mathcal{T}), R)$ and $\mathrm{colim}_{\mathcal{I}}^{\mathrm{SH}} : \mathrm{FUNC}(\mathcal{I}, \mathrm{SH}((\mathcal{C}, \mathcal{T}), R)) \to \mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$ are exact.

Proof: We know that $\mathrm{PSH}((\mathcal{C}, \mathcal{T}), R)$ is an abelian $\mathcal{U}$-category due to (**2.3.2**(c)) and we also know that $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$ is an abelian $\mathcal{U}$-category due to (**4.2.18**(a)).

For (a): As shown in (**3.2.10**), we have a projective generator $G = \coprod_{C \in \mathrm{Ob}(\mathcal{C})} R^{(C)}$ in $\mathrm{PSH}((\mathcal{C}, \mathcal{T}), R)$. We now have to show that $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$ has a generator: In (**3.2.10**) we also constructed an isomorphism $\mathrm{Hom}_{\mathrm{PSH}((\mathcal{C},\mathcal{T}),R)}(G, \_) \cong \prod_{C \in \mathrm{Ob}(\mathcal{C})} (\_)C$ in $\mathrm{FUNC}(\mathrm{PSH}((\mathcal{C}, \mathcal{T}), R), \mathrm{AB}_{\mathcal{U}})$. Furthermore, with the adjoint pair $((\_)^{\mathrm{sh}}, \iota)$ from (**4.2.16**), we have the isomorphisms $\mathrm{Hom}_{\mathrm{SH}((\mathcal{C},\mathcal{T}),R)}(G^{\mathrm{sh}}, \_) \cong \mathrm{Hom}_{\mathrm{PSH}((\mathcal{C},\mathcal{T}),R)}(G, \iota(\_)) \cong \prod_{C \in \mathrm{Ob}(\mathcal{C})} \iota(\_)C \cong \prod_{C \in \mathrm{Ob}(\mathcal{C})} (\_)C$ in $\mathrm{FUNC}(\mathrm{SH}((\mathcal{C}, \mathcal{T}), R), \mathrm{AB}_{\mathcal{U}})$. Analogously to (**3.2.10**), it is clear that $\prod_{C \in \mathrm{Ob}(\mathcal{C})} (\_)C$ is a faithful functor due to how morphisms of natural transformations are constructed, thus $\mathrm{Hom}_{\mathrm{SH}((\mathcal{C},\mathcal{T}),R)}(G^{\mathrm{sh}}, \_)$ is a faithful functor. Since $\mathrm{Hom}_{\mathrm{SH}((\mathcal{C},\mathcal{T}),R)}(G^{\mathrm{sh}}, \_)$ is left exact due to (**2.4.9**), we have that $\mathrm{Hom}_{\mathrm{SH}((\mathcal{C},\mathcal{T}),R)}(G^{\mathrm{sh}}, \_)$ is conservative due to (**3.2.4**(a)) and therefore G^{sh} is a generator of $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$.

For (b): We already know that $\mathrm{PSH}((\mathcal{C}, \mathcal{T}), R)$ has all $\mathcal{U}$-small colimits, since ${}_R\mathrm{MOD}_{\mathcal{U}}$ has all $\mathcal{U}$-small limits and colimits due to (**1.5.12**). Since ${}_R\mathrm{MOD}_{\mathcal{U}}$ is a good concrete $\mathcal{U}$-category due to (**4.1.2**(c)), it follows from (**4.2.17**(b)) that $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$ has all $\mathcal{U}$-small colimits.

For (c): Since filtered exact colimits are exact in ${}_R\mathrm{MOD}_{\mathcal{U}}$ due to (**2.4.12**), it is also clear that filtered exact colimits are exact in $\mathrm{PSH}((\mathcal{C}, \mathcal{T}), R)$ due to (**I.5.3.1** [Mor20]).

Colimits in $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$ are right exact since $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$ is an abelian $\mathcal{U}$-category and (**2.4.11**). It therefore suffices to show that filtered $\mathcal{U}$-small colimits are left exact in $\mathrm{SH}((\mathcal{C}, \mathcal{T}), R)$: Let $\iota : \mathrm{SH}((\mathcal{C}, \mathcal{T}), R) \to \mathrm{PSH}((\mathcal{C}, \mathcal{T}), R)$ be the inclusion functor which is left exact due to (**4.2.18**(b)). ι induces a functor $(\iota \circ \_) : \mathrm{FUNC}(\mathcal{I}, \mathrm{SH}((\mathcal{C}, \mathcal{T}), R)) \to \mathrm{FUNC}(\mathcal{I}, \mathrm{PSH}((\mathcal{C}, \mathcal{T}), R))$ that maps diagrams D to $\iota \circ D$, which is left exact due to (**I.5.3.1** [Mor20]). Since the colimit functor $\mathrm{colim}_{\mathcal{I}}^{\mathrm{PSH}}$ and sheafification $(\_)^{\mathrm{sh}}$ are exact, the colimit functor $\mathrm{colim}_{\mathcal{I}}^{\mathrm{SH}}$ is left exact since $\mathrm{colim}_{\mathcal{I}}^{\mathrm{SH}} = (\_)^{\mathrm{sh}} \circ \mathrm{colim}_{\mathcal{I}}^{\mathrm{PSH}} \circ (\iota \circ \_)$ is the composition of three left exact functors and therefore $\mathrm{colim}_{\mathcal{I}}^{\mathrm{SH}}$ is exact as claimed. ☐

4.3 Grothendieck Pretopologies on Abelian Categories

In this section, we will see an important explicit example of Grothendieck pretopologies by defining them on any abelian $\mathcal{U}$-category $\mathcal{A}$. Furthermore, we will find a useful characterization for sheaves in these Grothendieck pretopologies and show that the Yoneda embedding factorizes through these sheaves. These results will all be directly applied in *Mitchell's embedding theorem.* As a reminder, we have a Grothendieck universe $\mathcal{U}$ and a good concrete $\mathcal{U}$-category $\mathcal{D}$. We let $\mathcal{A}$ be an abelian $\mathcal{U}$-category.

4.3.1 Example (Sheaves on Abelian Categories) (A.3.6 [Mor20]**):** As $\mathcal{A}$ contains all finite limits due to (**2.3.5**), $\mathcal{A}$ contains all fiber products. The *canonical pretopology on* $\mathcal{A}$ is the $\mathcal{U}$-Grothendieck pretopology $\mathcal{T}_{\text{can}}$ where for every object A in $\mathcal{A}$, the covering families of $\mathcal{A}$ are exactly of the form $\mathfrak{U} = (u : B \to A)$, whereby u is an epimorphism in $\mathcal{A}$. We will check that $(\mathcal{A}, \mathcal{T}_{\text{can}})$ defines a $\mathcal{U}$-Grothendieck pretopology:

(i) *Base Changes:* Let A be an object in $\mathcal{A}$ and $\mathfrak{A} = (u : B \to A)$ be a covering family of A, i.e. $u : B \to A$ is an epimorphism. Let $f : C \to A$ be an morphism and $(B \times_A C, p_B, p_C)$ be the fiber product given via:

$$\begin{array}{ccc} B \times_A C & \xrightarrow{p_B} & B \\ {\scriptstyle p_C}\downarrow & & \downarrow{\scriptstyle u} \\ C & \xrightarrow{f} & A \end{array} .$$

Due to (**2.6.3**(b)) and the fact that u is an epimorphism, p_C must also be an epimorphism, thus $f^*\mathfrak{A} = (p_C : B \times_A C \to C)$ is a covering family of C and $\mathcal{T}_{\text{can}}$ contains base changes.

(ii) *Compositions:* Let $\mathfrak{A} = (u : B \to A)$ be any covering family of A and $\mathfrak{B} = (v : C \to B)$ be any covering family of B, we have that u and v are epimorphisms. It is clear that $u \circ v$ is also an epimorphism, and thus $\mathfrak{A}' = (u \circ v : C \to A)$ is a covering family of A.

(iii) *Isomorphisms:* If $u : B \to A$ is an isomorphism in $\mathcal{A}$, u is in particular an epimorphism and thus $\mathfrak{A} = (u : B \to A)$ is a covering family of A.

Furthermore, it is clear that $(\mathcal{A}, \mathcal{T}_{\text{can}})$ is a $\mathcal{U}$-Grothendieck pretopology with $\mathcal{U}$-small covering families, as all covering families are singletons. Thus, many of the statements proven in **Section 4.2** are applicable to $(\mathcal{A}, \mathcal{T}_{\text{can}})$.

As $(\mathcal{A}, \mathcal{T}_{\text{can}})$ is a $\mathcal{U}$-Grothendieck pretopology, we can use (**4.2.2**) to define *$\mathcal{D}$-valued sheaves on abelian $\mathcal{U}$-categories* $\mathcal{A}$ via $\text{Sh}(\mathcal{A}, \mathcal{D}) = \text{Sh}((\mathcal{A}, \mathcal{T}_{\text{can}}), \mathcal{D})$, $\text{Sh}(\mathcal{A}, R) = \text{Sh}((\mathcal{A}, \mathcal{T}_{\text{can}}), R)$, $\text{Sh}(\mathcal{A}) = \text{Sh}((\mathcal{A}, \mathcal{T}_{\text{can}}))$ for a $\mathcal{U}$-ring R.

From now on, $(\mathcal{A}, \mathcal{T}_{\text{can}})$ will always denote the canonical $\mathcal{U}$-Grothendieck pretopology on $\mathcal{A}$.

4.3.2 Note (Characterizations of Sheaves) (A.3.6 [Mor20]**):** Let F be a presheaf in $\text{PSh}(\mathcal{A}, \mathcal{D})$. For all epimorphisms $f : A \to B$ in $\mathcal{A}$, i.e. covering families $\mathfrak{B} = (f : A \to B)$, we have the following fiber product induced from $f : A \to B$:

$$\begin{array}{ccc} A \times_B A & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array} .$$

We have per definition that $\mathcal{H}_B(\mathfrak{B}, F) = \text{Eq}(Fp_1, Fp_2) = \text{Ker}(Fp_1 - Fp_2)$. Therefore F is a sheaf if and only if for all objects B in $\mathcal{A}$ and all covering families $\mathfrak{B} = (f : A \to B)$, we have that $\iota_{B,\mathfrak{B}} : FB \to \mathcal{H}_B(\mathfrak{B}, F)$, $a \mapsto Ff(a)$ is an isomorphism, due to (**4.2.3**). This is also equivalent to the following two statements:

(i) The separation morphism $\iota_{B,\mathfrak{B}} = Ff : FB \to FA$, as seen in (**4.2.2**), is a monomorphism.

(ii) The image of $\iota_{B,\mathfrak{B}} = Ff$ is isomorphic to $\mathcal{H}_B(\mathfrak{B}, F) = \text{Ker}(Fp_1 - Fp_2)$.

Combined, these two conditions are the same as checking that the following sequence is exact:

$$0 \longrightarrow FB \xrightarrow{Ff} FA \xrightarrow{Fp_1 - Fp_2} F(A \times_B A).$$

4.3.3 Note (Categories of Sheaves are Abelian): Since $(\mathcal{A}, \mathcal{T}_{\text{can}})$ is a $\mathcal{U}$-Grothendieck pretopology with $\mathcal{U}$-small covering families, we have that $\text{SH}(\mathcal{A}, \text{AB}_{\mathcal{U}})$ is an abelian category due to (**4.2.18**(a)).

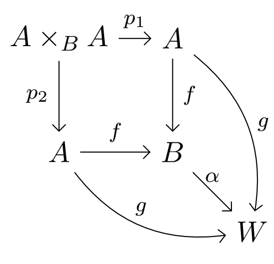
4.3.4 Note (Yoneda Embeddings for Abelian Categories) (A.4.3(b) [Mor20]**):** For construction of h': We claim that the Yoneda embedding $h : \mathcal{A} \to \text{PSH}(\mathcal{A})$ from (**1.4.6**) factors through $\text{PSH}(\mathcal{A}, \text{AB}_{\mathcal{U}})$, i.e. there exists an embedding $h' : \mathcal{A} \to \text{PSH}(\mathcal{A}, \text{AB}_{\mathcal{U}})$ such that h' composed with the inclusion $\iota : \text{PSH}(\mathcal{A}, \text{AB}_{\mathcal{U}}) \to \text{PSH}(\mathcal{A})$ is h. This is due to the fact that for all objects A in $\mathcal{A}$, h maps to the functor $h_A : \mathcal{A}^{\text{op}} \to \text{SET}_{\mathcal{U}}$, $B \mapsto \text{Hom}_{\mathcal{A}}(B, A)$, $f \mapsto f^*$ where we have canonical $\mathcal{U}$-group structures on $\text{Hom}_{\mathcal{A}}(B, A)$ with group homomorphisms f^*, induced from $\mathcal{A}$ being an abelian $\mathcal{U}$-category. This allows us to induce a presheaf $h_A : \mathcal{A}^{\text{op}} \to \text{AB}_{\mathcal{U}}$, $B \mapsto \text{Hom}_{\mathcal{A}}(B, A)$, $f \mapsto f^*$ and thus we can induce the embedding $h' : \mathcal{A} \to \text{PSH}(\mathcal{A}, \text{AB}_{\mathcal{U}})$, with $A \mapsto h_A$ for objects and $f \mapsto h_f$ for morphisms, as claimed.

For properties of h': Since the Yoneda embedding h is fully faithful as a consequence of the *Yoneda lemma* due to (**1.4.7**), it is clear that h' is also fully faithful due to h' having the same underlying morphism mappings as h. Due to (**2.4.9**) and using the fact that limits of presheaves are determined object-wise due to (**I.5.3.1** [Mor20]), we see that for a finite diagram $D : \mathcal{I} \to \mathcal{A}$, we have $\lim_{\mathcal{I}} h' D = \lim_{i \in \text{Ob}(\mathcal{I})} \text{Hom}_{\mathcal{A}}(\_, D(i)) \cong \text{Hom}_{\mathcal{A}}(\_, \lim_{i \in \text{Ob}(\mathcal{I})} D(i)) = h'(\lim_{\mathcal{I}} D)$. This implies that h' commutes with finite limits and is thus left exact.

4.3.5 Lemma (Representable Presheaves are Sheaves) (A.3.6(c) [Mor20]**):** Let F be a presheaf in $\text{PSH}(\mathcal{A}, \text{AB}_{\mathcal{U}})$ that is *representable*, which is defined as follows: For the Yoneda embedding $h' : \mathcal{A} \to \text{PSH}(\mathcal{A}, \text{AB}_{\mathcal{U}})$ from (**4.3.4**), we have that there exists an object A in $\mathcal{A}$ such that $F \cong h_A = h'A$ in $\text{PSH}(\mathcal{A}, \text{AB}_{\mathcal{U}})$. We then claim that F is a sheaf.

Proof: Since F is representable, we have that $F \cong h_A \cong \text{Hom}_{\mathcal{A}}(\_, A)$ for an object A in $\mathcal{A}$. In (**2.4.9**), we learned that $\text{Hom}_{\mathcal{A}}(\_, A)$ is a left exact functor, which clearly also makes F left exact.

For any covering family $\mathfrak{B} = (f : A \to B)$ of an object B in $\mathcal{A}$, i.e. any epimorphism $f : A \to B$, let $(A \times_B A, p_1, p_2)$ be the fiber product generated from the morphism $f : A \to B$ twice. We then claim f is the cokernel of $p_1 - p_2 : A \times_B A \to A$. As f is an epimorphism, we have that p_1 and p_2 are epimorphisms due to (**2.6.3**(b)). With $u : A \times_B A \to A \oplus A$ and $v : A \oplus A \to B$ as defined in (**2.6.2**), it can be shown that v is an epimorphism due to the fact that f, p_1 and p_2 are epimorphisms. This makes $A \times_B A \xrightarrow{u} A \oplus A \xrightarrow{v} B \to 0$ into an exact sequence. Therefore, due to (**2.6.2**(b)), we have that the square in the following diagram:



,

is cocartesian as well as cartesian. Therefore for any morphism $g : A \to W$ in $\mathcal{A}$ such that $g \circ (p_1 - p_2)$ is the zero morphism, i.e. $g \circ p_1 = g \circ p_2$, we have that there exists exactly one morphism $\alpha : B \to W$ in $\mathcal{A}$ such that the above diagram commutes. This implies that f fulfills the cokernel property of $p_1 - p_2$.

We therefore know that the sequence $A \times_B A \xrightarrow{p_1 - p_2} A \xrightarrow{f} B \to 0$ is exact. Due to F being a contravariant left exact functor, we then have that the sequence $0 \to FB \xrightarrow{Ff} FA \xrightarrow{Fp_1 - Fp_2} F(A \times_B A)$ in $\text{AB}_{\mathcal{U}}$ is exact due to (**2.4.8**). Since the object B and covering family $\mathfrak{B} = (f : A \to B)$ were freely chosen, we have that F is a sheaf due to (**4.3.2**). ☐

4.3.6 Lemma (Yoneda Embeddings Factorize through Sheaves) (A.4.3(b),(c) [Mor20]**):** We have that the Yoneda embedding $h : \mathcal{A} \to \mathrm{PSh}(\mathcal{A})$ from (**1.4.6**) factors through $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$. In other words, there exists a functor $h'' : \mathcal{A} \to \mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$ such that $h = \iota \circ h''$ for the inclusion functor $\iota : \mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}}) \to \mathrm{PSh}(\mathcal{A})$. Furthermore, we claim that h'' is a fully faithful exact functor.

Proof: For existence of h'' and h'' being fully faithful: Due to (**4.3.4**), we have that the Yoneda embedding h factors through $\mathrm{PSh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$, therefore it is enough to show that the Yoneda embedding $h' : \mathcal{A} \to \mathrm{PSh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$ factors through $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$. We define $h'' : \mathcal{A} \to \mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$ through the object mapping $A \mapsto h_A = \mathrm{Hom}_{\mathcal{A}}(\_, A)$. Since for all objects A in $\mathcal{A}$, we have that $h_A : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}_{\mathcal{U}}$ is by definition representable, it follows from (**4.3.5**) that h_A is a sheaf and lies in $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$. Furthermore, we define the morphism mappings of h'' exactly as they are defined in h and h', i.e. $f \mapsto h_f$. Since h and h' are fully faithful due to (**1.4.7**) and (**4.3.4**) and since $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$ is a full subcategory of $\mathrm{PSh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$, we have that h'' is fully faithful.

For h'' being left exact: The functors $(\_)^{\mathrm{sh}}$ and h' are exact due to (**4.2.18**(b)) and (**4.3.4**), which implies that $(\_)^{\mathrm{sh}} \circ h' : \mathcal{A} \to \mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$ is left exact as the composition of two left exact functors. Since h' already maps objects in $\mathcal{A}$ to sheaves in $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$, it is clear that $(\_)^{\mathrm{sh}} \circ h'$ is naturally isomorphic to h'' with the help of (**4.2.14**(e)). Therefore, h'' is left exact since $(\_)^{\mathrm{sh}} \circ h'$ is left exact.

For h'' being right exact: We denote images in $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$ with $\mathrm{Im}^{\mathrm{sh}}$ and images in $\mathrm{PSh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$ with Im. To show that h'' is right exact and thus exact, it is clearly enough to show that h'' maps epimorphisms to epimorphisms due to (**2.4.8**). Let $f : A \to B$ be an epimorphism in $\mathcal{A}$, we want to show that $h_f : h_A \to h_B$ is an epimorphism, or equivalently that $\mathrm{Im}^{\mathrm{sh}}(h_f) \cong (\mathrm{Im}(h_f))^{\mathrm{sh}}$ is isomorphic to h_B. The natural transformation h_f induces the morphisms $(f_* = h_f W : \mathrm{Hom}_{\mathcal{A}}(W, A) \to \mathrm{Hom}_{\mathcal{A}}(W, B))_{W \in \mathrm{Ob}(\mathcal{A})}$, and the canonical monomorphism $p : \mathrm{Im}(h_f) \to B$ in $\mathrm{PSh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$ with the inclusion monomorphisms $(pW : \mathrm{Im}(h_f)W \to \mathrm{Hom}_{\mathcal{A}}(W, B))_{W \in \mathrm{Ob}(\mathcal{A})}$. We know that the image morphism of sheaves $p^{\mathrm{sh}} : \mathrm{Im}^{\mathrm{sh}}(hf) \to h_B$ is given by the sheafification $((pW)^{\mathrm{sh}})_{W \in \mathrm{Ob}(W)}$ due to the proof of (**4.2.18**). Let W be an object in $\mathcal{A}$, due to the construction of the separation functor in (**4.2.12**($\star\star$)), we have that $p^{\mathrm{sep}}W = \mathrm{colim}_{\mathrm{Cov}^0(W)^{\mathrm{op}}} h_f W_{\mathcal{H}} : \mathrm{Im}(h_f)^{\mathrm{sep}}W \to h_B^{\mathrm{sep}}W \cong h_B W$, with $h_f W_{\mathcal{H}}$ as defined in (**4.2.12**($\star\star$)). Since $(\_)^{\mathrm{sh}} = (\_)^{\mathrm{sep}} \circ (\_)^{\mathrm{sep}}$ where $(\_)^{\mathrm{sep}}$ is a left exact functor due to the proof of (**4.2.17**(d)), and since colimit functors map epimorphisms to epimorphisms due to (**2.4.11**) and (**2.4.8**), it is enough to show that the monomorphism $(pW)^{\mathrm{sep}}$ is an isomorphism for every object W in $\mathcal{A}$.

Since h_A and h_B are sheaves due to (**4.3.5**), we have for any object W in $\mathcal{A}$ and for any covering family $\mathfrak{W}$ of W the isomorphisms $\mathcal{H}_W(\mathfrak{W}, h_A) \cong \mathrm{Hom}_{\mathcal{A}}(W, A)$ and $\mathcal{H}_W(\mathfrak{W}, h_B) \cong \mathrm{Hom}_{\mathcal{A}}(W, B)$. Due to the construction of filtered colimits in $\mathrm{Ab}_{\mathcal{U}}$ with the help of (**1.5.16**) and (**1.5.17**(b)), it is enough to show that for every element $u \in \mathrm{Hom}_{\mathcal{A}}(W, B)$, there exists a covering family $\mathfrak{W} = (g : G \to W)$ of W such that for the element $h_B g(u) = g^*(u) = u \circ g \in \mathcal{H}_W(\mathfrak{W}, h_B) \cong \mathrm{Hom}_{\mathcal{A}}(G, B)$, there exists an element $k \in \mathcal{H}_W(\mathfrak{W}, h_A) \cong \mathrm{Hom}_{\mathcal{A}}(G, A)$ such that $h_f W_{\mathfrak{W}\mathcal{H}}(k) = h_f(k) = h_B g(u)$. This claim would imply that u lies within $\mathrm{Im}^{\mathrm{sep}}(h_f)W \cong (\mathrm{Im}(h_f))^{\mathrm{sep}}W$ and that the monomorphisms $p^{\mathrm{sep}}W$ and $p^{\mathrm{sh}}W$ are isomorphisms. Let $u \in h_B(W) = \mathrm{Hom}_{\mathcal{A}}(W, B)$ and observe the fiber product with the projections p_A and p_W:

$$\begin{array}{ccc} A \times_B W & \xrightarrow{p_W} & W \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle u} \\ A & \xrightarrow{f} & B \end{array},$$

since f is an epimorphism, p_W is also an epimorphism due to (**2.6.3**(b)), this makes $(p_W) = \mathfrak{W}$ a covering family of W in $(\mathcal{A}, \mathcal{T}_{\mathrm{can}})$. Due to the commutativity of the diagram, we have $h_f(p_A) = f \circ p_A = u \circ p_W = h_B p_W(u)$. Thus, when we set $g = p_W$ and $k = p_A$, the claim follows and h_f is an epimorphism in $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_{\mathcal{U}})$. $\square$

5 The Mitchell Embedding Theorem

Now we have all the results required to state and properly understand the proof of Mitchell's embedding theorem. Let $\mathcal{U}$ be a Grothendieck universe.

5.1 Theorem

The Mitchell Embedding Theorem (III.3.1 [Mor20]**),** ([NCa19])**:** Let $\mathcal{A}$ be a $\mathcal{U}$-small abelian category. There exists a $\mathcal{U}$-ring R with a fully faithful exact functor $G : \mathcal{A} \to {}_R\mathrm{Mod}_\mathcal{U}$ to the $\mathcal{U}$-left-R-modules. Analogously there exists a $\mathcal{U}$-ring R and a fully faithful exact functor $G : \mathcal{A} \to \mathrm{Mod}_{R\mathcal{U}}$ to the $\mathcal{U}$-right-R-modules.

Proof: Due to (**1.3.9**), it is clear that it is enough to show the existence of $G : \mathcal{A} \to {}_R\mathrm{Mod}_\mathcal{U}$.

We denote the canonical $\mathcal{U}$-small Grothendieck pretopology on $\mathcal{A}$ by $(\mathcal{A}, \mathcal{T}_{\text{can}})$, then due to (**4.3.6**), we have that the Yoneda embedding from (**1.4.6**), denoted by $h : \mathcal{A} \to \mathrm{Set}_\mathcal{U}$, factorizes through the category of sheaves $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U}) = \mathrm{Sh}((\mathcal{A}, \mathcal{T}_{\text{can}}), \mathrm{Ab}_\mathcal{U})$. More precisely, for the canonical inclusion functor $\iota : \mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U}) \to \mathrm{PSh}(\mathcal{A})$, there exists an embedding functor $h'' : \mathcal{A} \to \mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})$ such that $h = \iota \circ h''$. (**4.3.6**) also implies that h'' is a fully faithful exact functor.

Since $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})$ is a Grothendieck abelian $\mathcal{U}$-category due to (**4.2.19**) and $\mathcal{A}$ being $\mathcal{U}$-small, $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})$ has an injective cogenerator G due to (**3.4.7**). Furthermore, since $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})$ is a Grothendieck abelian $\mathcal{U}$-category, it is an abelian $\mathcal{U}$-category and it has all $\mathcal{U}$-small colimits. The above statements imply due to duality that $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})^{\text{op}}$ has all $\mathcal{U}$-small limits and is an abelian $\mathcal{U}$-category as seen in (**2.3.3**), and furthermore has a projective generator G.

For the embedding $h'' : \mathcal{A} \to \mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})$, we observe the image $h''(\mathcal{A})$ as a subgraph of $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})$. Due to h'' being fully faithful, it is easy to check that $h''(\mathcal{A})$ is a well-defined full subcategory of $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})$. Since $\mathcal{A}$ is $\mathcal{U}$-small, it is clear that $h''(\mathcal{A})$ is also $\mathcal{U}$-small. Due to h'' being exact, h'' commutes with finite limits and colimits, and due to $\mathcal{A}$ having all finite limits and colimits due to (**2.3.5**), it is clear that $h''(\mathcal{A})$ is stable under all finite limits and colimits of $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})$, as defined in (**3.3.2**).

Due to duality, it is clear that $h''^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})^{\text{op}}$ is also a fully faithful exact functor, with $h''(\mathcal{A})^{\text{op}}$ being a $\mathcal{U}$-small full subcategory of $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})^{\text{op}}$ which is stable under all finite limits and colimits of $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})^{\text{op}}$. Since $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})^{\text{op}}$ has a projective generator G, all the prerequisites for (**3.3.3**(b)) are fulfilled, which then implies that there exists an object H in $\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})^{\text{op}}$, a $\mathcal{U}$-ring $S = \mathrm{Hom}_{\mathrm{Sh}(\mathcal{A}, \mathrm{Ab}_\mathcal{U})^{\text{op}}}(H, H)$ of endomorphisms and a fully faithful exact functor $F : h''(\mathcal{A})^{\text{op}} \to \mathrm{Mod}_{S\mathcal{U}}$. Due to the Yoneda embedding h'' being fully faithful and exact, we have that $F \circ h'' : \mathcal{A}^{\text{op}} \to \mathrm{Mod}_{S\mathcal{U}}$ is a fully faithful exact functor. Due to (**1.3.9**), we have the category-equivalence $\mathrm{Mod}_{S\mathcal{U}} \cong {}_{S^{\text{op}}}\mathrm{Mod}_\mathcal{U}$, which induces a fully faithful exact functor $G : \mathcal{A}^{\text{op}} \to {}_{S^{\text{op}}}\mathrm{Mod}_\mathcal{U}$ from $F \circ h''$.

If we apply the argument above, but starting with $\mathcal{A}^{\text{op}}$ instead of $\mathcal{A}$, which is still a $\mathcal{U}$-small abelian category due to (**2.3.3**), we would instead receive a fully faithful exact functor $G : \mathcal{A} \to {}_R\mathrm{Mod}_\mathcal{U}$ for a $\mathcal{U}$-ring R as desired. □

5.2 Applications of the Mitchell Embedding Theorem

Let $\mathcal{A}$ be an abelian $\mathcal{U}$-category that is not necessarily $\mathcal{U}$-small. Although *Mitchell's embedding theorem* may not be used on $\mathcal{A}$ since it may not be $\mathcal{U}$-small, we can extend its usability to $\mathcal{U}$-small diagrams $D : \mathcal{I} \to \mathcal{A}$, by claiming that D factorizes through a $\mathcal{U}$-small subcategory $\mathcal{B}$ of $\mathcal{A}$ where *Mitchell's embedding theorem* can be applied on $\mathcal{B}$. This would enable us to apply diagram chasing, as mentioned in the **Motivation and Introduction**, onto abelian $\mathcal{U}$-categories $\mathcal{A}$.

5.2.1 Lemma ($\mathcal{U}$-Small Abelian Subcategories) ([Sta14])**, (1.3.2** [Wei94]**):** Let $\mathcal{A}$ be an abelian $\mathcal{U}$-category and $D : \mathcal{I} \to \mathcal{A}$ be a $\mathcal{U}$-small diagram. Then the subgraph $D(\mathcal{I})$ of $\mathcal{A}$, i.e. the image of D in $\mathcal{A}$, is contained within an abelian full $\mathcal{U}$-small subcategory $\mathcal{B}$ of $\mathcal{A}$. Furthermore, the inclusion functor $\iota : \mathcal{B} \to \mathcal{A}$ is fully faithful and exact.

Proof: For construction of $\mathcal{B}_0$: As $D(\mathcal{I})$ is a subgraph of $\mathcal{A}$, we claim that there exists a minimal full $\mathcal{U}$-small subcategory of $\mathcal{B}_0$ of $\mathcal{A}$ with the following properties: $\mathcal{B}_0$ contains all objects and morphisms of $D(\mathcal{I})$ and $\mathcal{B}_0$ contains all finite biproducts of objects in $D(\mathcal{I})$ as they exist in $\mathcal{A}$, as seen in (**2.3.1**). $\mathcal{B}_0$ can be shown to exist through explicit construction: Let $\mathrm{Ob}(\mathcal{B}_0)$ be the set of all objects A of $\mathcal{A}$ where there exists a finite collection of objects $(D(i_j))_{j\in J}$ in $D(\mathcal{I})$ such that $A \cong \bigoplus_{j\in J} D(i_j)$. Let the morphisms $\mathrm{Mor}(\mathcal{B}_0)$ be all the morphisms f in $\mathcal{A}$ such that the domains and codomains are objects in $\mathcal{B}_0$, i.e. $\mathrm{dom}(f), \mathrm{cod}(f) \in \mathrm{Ob}(\mathcal{B}_0)$.

It is easy to check that this construction forms a full subcategory $\mathcal{B}_0$ of $\mathcal{A}$, which implies directly that $\mathcal{B}_0$ is a $\mathcal{U}$-category as $\mathcal{A}$ is a $\mathcal{U}$-category. Furthermore, we claim that $\mathcal{B}_0$ is $\mathcal{U}$-small: Since $\mathrm{Ob}(D(\mathcal{I}))$ is $\mathcal{U}$-small due to $\mathcal{I}$ being $\mathcal{U}$-small, we have that $\prod_{n\in\mathbb{N}_0} \mathrm{Ob}(D(\mathcal{I}))$ is $\mathcal{U}$-small due to (**I.1.3**(viii) [Mor20]). Since the cardinality of $\mathrm{Ob}(\mathcal{B}_0)$ is clearly lesser or equal to the cardinality of the $\mathcal{U}$-small set $\prod_{n\in\mathbb{N}_0} \mathrm{Ob}(D(\mathcal{I}))$, $\mathrm{Ob}(\mathcal{B}_0)$ is itself $\mathcal{U}$-small with the help of (**1.1.3**(c)).

For the minimality property of $\mathcal{B}_0$, let $\mathcal{A}$ be a full $\mathcal{U}$-small subcategory of $\mathcal{A}$ that contains $D(\mathcal{I})$ and the corresponding finite coproducts of objects in $D(\mathcal{I})$. It is clear that by construction, $\mathcal{A}$ must contain $\mathcal{B}_0$.

For construction of $\mathcal{B}_{n+1}$ given $\mathcal{B}_n$, $n \in \mathbb{N}_0$: For all $n \in \mathbb{N}_0$, we want to recursively define a full $\mathcal{U}$-small subcategory $\mathcal{B}_{n+1}$ of $\mathcal{A}$ that contains $D(\mathcal{I})$ and the corresponding biproducts of $D(\mathcal{I})$, under the assumption that $\mathcal{B}_n$ also has these properties. We have already defined the base case $\mathcal{B}_0$.

Given a full $\mathcal{U}$-small subcategory $\mathcal{B}_n$ that contains $D(\mathcal{I})$ and the corresponding finite biproducts, we construct $\mathcal{B}_{n+1}$ as follows: $\mathcal{B}_{n+1}$ contains all objects and morphisms from $\mathcal{B}_n$, i.e. $\mathcal{B}_n$ is a subcategory of $\mathcal{B}_{n+1}$ and thus $D(\mathcal{I})$ is a subcategory of $\mathcal{B}_{n+1}$. We define $\mathrm{Ob}(\mathcal{B}_{n+1})$ as follows: Let $\mathrm{Ob}(\mathcal{B}_{n+1})$ be all objects A of $\mathcal{A}$ where there exists a finite collection of objects $(A_j)_j \in J$ for which each A_j lies in $\mathrm{Ob}(\mathcal{B}_n)$ or is a kernel or cokernel of a morphism in $\mathcal{B}_n$, such that $A \cong \bigoplus_{j\in J} A_j$. For $\mathrm{Mor}(\mathcal{B}_{n+1})$, add all morphisms f in $\mathcal{A}$ such that $\mathrm{dom}(f), \mathrm{cod}(f) \in \mathrm{Ob}(\mathcal{B}_{n+1})$. This makes $\mathcal{B}_{n+1}$ into a full subcategory.

$\mathcal{B}_{n+1}$ is still $\mathcal{U}$-small, as $\mathrm{Ob}(\mathcal{B}_{n+1})$ has at most the same cardinality as $\prod_{n\in\mathbb{N}_0} x$, whereby x is a set with the cardinality of $\mathrm{Ob}(\mathcal{B}_n)$ in disjoint union with two disjoint copies of $(\mathrm{Hom}_{\mathcal{A}}(A,B))_{A,B\in\mathrm{Ob}(\mathcal{B}_n)}$ (one for the kernels and one for the cokernels). It is easy to check that x is $\mathcal{U}$-small and thus that $\prod_{n\in\mathbb{N}_0} x$ is $\mathcal{U}$-small, with the help of (**1.1.3**(c)) we have that $\mathrm{Ob}(\mathcal{B}_{n+1})$ is $\mathcal{U}$-small.

For construction of $\mathcal{B}$: Take $\mathcal{B} = \bigcup_{n\in\mathbb{N}_0} \mathcal{B}_n$ to be the smallest full subcategory of $\mathcal{A}$ that contains all full subcategories of the form $\mathcal{B}_n$. $\mathcal{B}$ exists with a similar argument to the existence and minimality of $\mathcal{B}_0$, as it is the unique full subcategory of $\mathcal{A}$ whereby $\mathrm{Ob}(\mathcal{B}) = \bigcup_{n\in\mathbb{N}_0} \mathrm{Ob}(\mathcal{B}_n)$. $\mathcal{B}$ clearly contains $D(\mathcal{I})$ and has a bilinear composition $\circ$. Furthermore, $\mathcal{B}$ contains the kernels and cokernels of its morphisms, since every morphism f in $\mathcal{B}$ is a morphism of $\mathcal{B}_n$ for some $n \in \mathbb{N}_0$, and thus its kernel and cokernel lies in $\mathcal{B}_{n+1} \subset \mathcal{B}$. Analogously, $\mathcal{B}$ contains all finite biproducts, including a zero object 0. Since the decompositions of morphisms $f \in \mathrm{Mor}(\mathcal{B})$ from (**2.2.5**) are the same in $\mathcal{B}$ than that of $\mathcal{A}$, the induced morphism $u_f : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ in $\mathcal{B}$ is an isomorphism, making $\mathcal{B}$ an abelian $\mathcal{U}$-category. As each $\mathrm{Ob}(\mathcal{B}_n)$ is $\mathcal{U}$-small, it follows that $\mathrm{Ob}(\mathcal{B}) = \bigcup_{n\in\mathbb{N}_0} \mathrm{Ob}(\mathcal{B}_n)$ is $\mathcal{U}$-small due to (**1.1.2**).

For exactness of inclusion functor being fully faithful and exact: ι is fully faithful due to $\mathcal{B}$ being a full subcategory of $\mathcal{A}$. Using (**2.4.8**), it is enough to prove that ι commutes with kernels and cokernels. Due to our construction of $\mathcal{B}$, the kernel and cokernel in $\mathcal{A}$ of every morphism $f \in \mathrm{Mor}(\mathcal{B})$ also lies in $\mathcal{B}$, which fulfills the properties of kernels and cokernels in $\mathcal{B}$. Thus we have $\mathrm{Ker}(\iota f) \cong \iota\mathrm{Ker}(f)$ and $\mathrm{Coker}(\iota f) \cong \iota\mathrm{Coker}(f)$ which implies the exactness of ι. □

5.2.2 Applications (The Mitchell Embedding Theorem on Large Categories): The most famous applications of *Mitchell's embedding theorem* in (**5.1**), used together with (**5.2.1**), are to prove generalizations of important lemmas in homological algebra that have been proven by diagram chasing. Here is a brief overview of such lemmas and how *Mitchell's embedding theorem* can be applied on them:

(a) **Short Five Lemma for Modules (2.2** [Vit10]**):** Let R be a $\mathcal{U}$-ring, observe the following commutative diagram in ${}_R\text{MOD}_{\mathcal{U}}$ such that the rows are exact:

(⋆)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array},$$

then the following statements are true:

(a) If α and γ are monomorphisms, then β is a monomorphism.

(b) If α and γ are epimorphisms, then β is an epimorphism.

(c) If α and γ are isomorphisms, β is an isomorphism.

Proof: See reference for the case where the diagram lies in $\text{AB}_{\mathcal{U}} \cong {}_{\mathbb{Z}}\text{MOD}_{\mathcal{U}}$. The proof in (**2.2** [Vit10]) generalizes without significant modification to ${}_R\text{MOD}_{\mathcal{U}}$. □

(b) **Short Five Lemma for Abelian Categories:** Let $\mathcal{A}$ be any abelian $\mathcal{U}$-category, then we claim that the short five lemma applies as well for all diagrams of the form (⋆) in $\mathcal{A}$.

Proof: Any such diagram in $\mathcal{A}$ clearly defines a $\mathcal{U}$-small subgraph of $\mathcal{A}$ as it is finite. This subgraph induces a $\mathcal{U}$-small category $\mathcal{I}$, which is the smallest subcategory of $\mathcal{A}$ that contains the original diagram of the form (⋆). With the canonical inclusion functor $D : \mathcal{I} \to \mathcal{A}$ and (**5.2.1**), there exists an abelian full $\mathcal{U}$-small subcategory $\mathcal{B}$ of $\mathcal{A}$ that contains $\mathcal{I}$ and therefore also the diagram of the form (⋆). Using *Mitchell's embedding theorem* from (**5.1**) implies that there exists a fully faithful exact functor $G : \mathcal{B} \to {}_R\text{MOD}_{\mathcal{U}}$ for a $\mathcal{U}$-ring R. Our diagram thus embeds itself into ${}_R\text{MOD}_{\mathcal{U}}$, where we can apply the short five lemma from (**5.2.2**(a)).

Since G and the inclusion $\iota : \mathcal{B} \to \mathcal{A}$ are exact, G, ι send monomorphisms (respectively epimorphisms and isomorphisms) to monomorphisms (respectively epimorphisms and isomorphisms) due to (**2.4.8**). Since G and ι are faithful and exact, G, ι reflect monomorphisms (respectively epimorphisms and isomorphisms) due to (**3.2.3**) and (**3.2.4**). Since the short five lemma in (**5.2.2**(a)) only implies statements on whether morphisms are monomorphisms, epimorphisms or isomorphisms, we conclude the following: The statements of the short five lemma from (**5.2.2**(a)), applied on the diagram of the form (⋆) embedded into ${}_R\text{MOD}_{\mathcal{U}}$ via $G : \mathcal{B} \to {}_R\text{MOD}_{\mathcal{U}}$, imply that the short five lemma holds in $\mathcal{B}$. We then have the generalization of the short five lemma from (**5.2.2**(a)) as it also applies in $\mathcal{A}$. This follows from the monomorphism- (epimorphism- and isomorphism-) preserving properties of G and ι. □

In summary, the short five lemma, through the *Mitchell embedding theorem* from (**5.1**) and (**5.2.1**) can be generalized to work for any commutative diagram of the form (⋆) in any abelian $\mathcal{U}$-category $\mathcal{A}$.

For the following diagram chasing lemmas, the application of *Mitchell's embedding theorem* is completely analogous to that of the short five lemma. This is because the statements of these lemmas involve $\mathcal{U}$-small diagrams, the existence of morphisms and whether they are monomorphisms, epimorphisms or isomorphisms. These statements are invariant under fully faithful exact functors and thus *Mitchell's embedding theorem* (**5.1**) and (**5.2.1**) are analogously applicable:

(c) **Five Lemma for Modules (1.3** [Hai18]**), (**[Pro19]**):** The five lemma is an important generalization of the short five lemma, which is often used to compute homologies and cohomologies of long exact sequences of modules and groups. It has almost the same diagram and setup as the short five lemma, but with the corners of the diagram in ($\star$) replaced with objects that are not necessarily zero objects. A statement of this lemma for modules and a proof through diagram chasing can be found in the references. *Mitchell's embedding theorem* from (**5.1**) with (**5.2.1**) generalizes this lemma to work for all abelian $\mathcal{U}$-categories.

(d) **Snake Lemma for Modules (1.14** [Hai18]**):** The snake lemma is a similarly important lemma to the five lemma, as it sometimes allows one to extend a collection of short exact sequences into a long exact sequence with the help of an induced "connecting morphism" between short exact sequences. A statement of the snake lemma for modules and a diagram chasing proof can be found in the reference. Applying *Mitchell's embedding theorem* (**5.1**) and (**5.2.1**) allows us to apply the snake lemma on all abelian $\mathcal{U}$-categories.

(e) **Snake Lemma for Abelian Categories:** The snake lemma for abelian $\mathcal{U}$-categories is useful for characterizing an important property of derived functors: Let $\mathcal{A}$ and $\mathcal{B}$ be abelian $\mathcal{U}$-categories such that $\mathcal{A}$ has enough injectives, as defined in (**3.1.6**(a)), then for an additive left exact functor $F : \mathcal{A} \to \mathcal{B}$, we define the *i-th right-derivative of F*, for $i \in \mathbb{N}_0$, as the functor $R^iF : \mathcal{A} \to \mathcal{B}$ seen in ([Wik21a]). Then for an exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, the exact sequence $0 \to FA \to FB \to FC$ in $\mathcal{B}$ can be extended with the help of the snake lemma to a long exact sequence:

$$0 \to FA \to FB \to FC \to R^1FA \to R^1FA \to R^1FB \to R^1FC \to R^2FA \to \dots,$$

whereby the morphisms $R^iFC \to R^{i+1}FA$ are the "connecting morphisms" induced by the snake lemma for all $i \in \mathbb{N}_0$.

(f) **Nine Lemma for Modules (2.4** [Hai18]**):** The nine lemma is a less used diagram chasing lemma than the other lemmas that can be derived from the snake lemma. A diagram chasing proof of the lemma for categories of modules may be found in the reference, which can be generalized to also work for abelian categories through the help of *Mitchell's embedding theorem* (**5.1**) from and (**5.2.1**).

6 Conclusion

Explaining the mathematical machinery behind *Mitchell's embedding theorem* (**5.1**), especially from the basics, requires a sizable amount of work and preparation: The constructions of Grothendieck universes, categories, functors and their various types (e.g. adjoint functors, natural transformations) are an important basis from which we study limits and colimits, hom-functors, the *Yoneda Lemma* (**1.4.5**) and abelian $\mathcal{U}$-categories. Obviously as *Mitchell's embedding theorem* (**5.1**) aims to construct a profound statement on abelian $\mathcal{U}$-categories, much of our focus will be on constructions on abelian $\mathcal{U}$-categories: We constructed injectives, projectives and Grothendieck abelian $\mathcal{U}$-categories (**3.4.1**), which have enough injectives and projectives due to (**3.4.6**). Sheaves and sheafifications are then introduced at a rather abstract level, due to our required construction of a $\mathcal{U}$-Grothendieck pretopology. This provided categories where the Yoneda embedding could factorize to give us the embedding in *Mitchell's embedding theorem* (**5.1**).

As much work is put into proving the theorem, it would make sense to ask how much insight can be gleaned from it. There are definitely uses stemming from generalizing certain lemmas proven for all categories of modules ${}_R\text{Mod}_{\mathcal{U}}$ and $\text{Mod}_{R\mathcal{U}}$, such that they also apply to abelian $\mathcal{U}$-categories, or at least $\mathcal{U}$-small abelian categories. With the help of the construction of (**5.2.1**), we saw how to apply *Mitchell's embedding theorem* (**5.1**) even when we do not have a $\mathcal{U}$-small abelian category $\mathcal{A}$ to work with. *Mitchell's embedding theorem* (**5.1**) is also likely very useful when aiming to quickly solve questions in abelian $\mathcal{U}$-categories by allowing mathematicians to rely on their intuitions with element- and set-based arguments allowed within modules, which are sometimes unavailable or cumbersome in category theory.

However, some mathematicians do have criticisms of *Mitchell's embedding theorem* (**5.1**): For example, *Martin Brandenburg* has stated in his commentary ([Sta17]) that especially when proving diagram chasing lemmas as in (**5.2.2**), there exists more elementary, category-theoretic proofs. These more direct proofs may be more elegant since they do not consider the category of modules as a special case that needs to be proven separately.

Regardless of any pitfalls the theorem has in its usefulness, proving the theorem is definitely an excellent exercise to familiarize oneself with category theory, sheaves and homological algebra, due to the variety of lemmas and theorems stemming from disparate constructions that one must learn. The *Mitchell embedding theorem* (**5.1**) provides a convenient and quick shortcut to proving certain statements for abelian $\mathcal{U}$-categories, even if some may consider other more involving and direct methods more elegant.

7 References

Books

[Bra16] Martin Brandenburg. *Einführung in die Kategorientheorie*. Springer, 2016. URL: https://www.springer.com/de/book/9783662470688.

[Fre64] Peter Freyd. *Abelian Categories*. Harper & Row, 1964. URL: https://www.maths.ed.ac.uk/~v1ranick/papers/freydab.pdf.

[JC21] Aise Johan de Jong and Collaborators. *The Stacks Project*. Open Source, 2021. URL: https://stacks.math.columbia.edu.

[Mit65] Barry Mitchell. *Theory of Categories*. Academic Press, 1965. URL: https://www.maths.ed.ac.uk/~v1ranick/papers/mitchell.pdf.

[SK06] Pierre Schapira and Masaki Kashiwara. *Categories and Sheaves*. Springer, 2006. URL: https://www.maths.ed.ac.uk/~v1ranick/papers/kashiwara2.pdf.

[Wei94] Charles Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/weibel-hom.pdf.

Manuscripts

[Aly08] Geillan Aly. *Abelian Categories and the Freyd-Mitchell Embedding Theorem*. Notes, 2008. URL: http://u.arizona.edu/~geillan/research/ab_categories.pdf.

[Bö20] Gebhard Böckle. *Algebra 2, Sommersemester 2020, Aufschrieb von Lucas Schmitt*. Lecture Notes, 2020. URL: https://drive.google.com/file/d/1-elHjGGks0yLNYTci5Gt9Jz4Z-2mnjrv/view.

[Bö21] Gebhard Böckle. *Algebraische Geometrie 1, Wintersemester 2020-2021, Vorlesung 17*. Lecture Notes, 2021. URL: https://drive.google.com/file/d/11WG_NKCOu6XxScj_z_GplB3xOYyhKGJf/view.

[Hai18] Peter Haine. *Diagram Chases*. Examples & Exercises, 2018. URL: https://math.mit.edu/~phaine/files/Diagram_Chases.pdf.

[Kie06] Andy Kiersz. *Colimits and Homological Algebra*. Notes, 2006. URL: http://www.math.uchicago.edu/~may/VIGRE/VIGRE2006/PAPERS/Kiersz.pdf.

[Lei14] Tom Leinster. *Basic Category Theory*. arXiv, 2014. URL: https://arxiv.org/pdf/1612.09375.pdf.

[Mor20] Sophie Morel. *Homological Algebra*. Notes, 2020. URL: http://perso.ens-lyon.fr/sophie.morel/notes540.pdf.

[RCT13] Thomas Rüd and Jean Claude-Ton. *Yoneda Lemma*. 2013. URL: https://personal.math.ubc.ca/~thomas/TeXthings/yonedalemmarapport(1).pdf.

[Sch21] Pierre Schapira. *Categories and Homological Algebra*. Notes, 2021. URL: https://perso.imj-prg.fr/pierre-schapira/wp-content/uploads/schapira-pub/lectnotes/HomAl.pdf.

[Vit10] Valentino Vito. *Some Notes on Diagram Chasing and Diagrammatic Proofs in Category Theory*. arXiv, 2010. URL: https://arxiv.org/pdf/2010.12534.pdf.

Websites

[Bel15] John Bell. *The Axiom of Choice: Maximal Principles and Zorn's Lemma*. 2015. URL: https://plato.stanford.edu/entries/axiom-choice/#MaxPriZorLem.

[Mat14] MathOverflow. *Can Epi/Mono for Natural Transformations be Checked Pointwise?* 2014. URL: https://mathoverflow.net/questions/17953/can-epi-mono-for-natural-transformations-be-checked-pointwise/17977#17977.

[Mat20] Encyclopedia of Mathematics. *Antinomy*. 2020. URL: https://encyclopediaofmath.org/wiki/Antinomy.

[Mat21] Encyclopedia of Mathematics. *Cantor Theorem*. 2021. URL: https://encyclopediaofmath.org/index.php?title=Cantor_theorem.

[NCa19] NCatlab. *Freyd-Mitchell Embedding Theorem*. 2019. URL: https://ncatlab.org/nlab/show/Freyd-Mitchell+embedding+theorem.

[NCa21a] NCatlab. *Baer's Criterion*. 2021. URL: https://ncatlab.org/nlab/show/Baer's+criterion.

[NCa21b] NCatlab. *Grothendieck Category*. 2021. URL: https://ncatlab.org/nlab/show/Grothendieck+category.

[NCa21c] NCatlab. *Grothendieck Pretopology*. 2021. URL: https://ncatlab.org/nlab/show/Grothendieck+pretopology.

[NCa21d] NCatlab. *Monomorphism*. 2021. URL: https://ncatlab.org/nlab/show/monomorphism.

[Pro19] ProofWiki. *Five Lemma*. 2019. URL: https://proofwiki.org/wiki/Five_Lemma.

[Sta14] StackExchange. *Applying Freyd-Mitchell's Embedding Theorem on Large Categories*. 2014. URL: https://math.stackexchange.com/questions/33196/applying-freyd-mitchells-embedding-theorem-on-large-categories.

[Sta15] StackExchange. *Right Module vs Left Module*. 2015. URL: https://math.stackexchange.com/questions/1493609/right-module-vs-left-module.

[Sta17] StackExchange. *Applications of Mitchell's Embedding Theorem*. 2017. URL: https://math.stackexchange.com/questions/361351/applications-of-mitchells-embedding-theorem/362159#362159.

[Sta21] StackExchange. *Hom-Tensor Adjunctions*. 2021. URL: https://math.stackexchange.com/questions/723368/hom-tensor-adjunctions.

[Wik21a] Wikipedia. *Derived Functor*. 2021. URL: https://en.wikipedia.org/wiki/Derived_functor.

[Wik21b] Wikipedia. *Limit (Category Theory)*. 2021. URL: https://en.wikipedia.org/wiki/Limit_(category_theory).

[Wik21c] Wikipedia. *Tarski–Grothendieck Set Theory*. 2021. URL: https://en.wikipedia.org/wiki/Tarski-Grothendieck_set_theory.

[Wik21d] Wikipedia. *von Neumann–Bernays–Gödel Set Theory*. 2021. URL: https://en.wikipedia.org/wiki/Von_Neumann-Bernays-GÃűdel_set_theory.

[Wik21e] Wikipedia. *Zermelo–Fraenkel Set Theory*. 2021. URL: https://en.wikipedia.org/wiki/Zermelo-Fraenkel_set_theory.